APPM 4/5560 Markov Chains

Fall 2019, Some Review Problems for the Final

1. Consider a two-server ququeing process where customers arrive according to a Poisson process with rate λ, and go to either one of two servers if they are available. If both servers are busy, the customers form a single queue where the person at the end of the line will go to the next available server.

   Suppose that each sever takes an exponential amount of time with rate μ to serve a customer.

   Find the probability that, in equilibrium, as most one server is busy.

2. Suppose that in an M/M/1 queue, the customer arrival rate is 3 per minute. Find the service rate so that 95% of the time the queue will contain less than 10 customers.

3. Sal the barber has a shop that can hold up to 3 customers, including the one being served. Excess customers get turned away. If customer interarrival times are exponential with rate λ, and Sal, as the only worker in the shop can serve people at a rate that is exponential also with rate λ, find the expected number of customers in the shop.

4. For an M/M/1 queue in equilibrium with interarrival rate λ and service rate μ, find the expected time between two consecutive times the queue is empty.

5. Consider a Poisson process $\{N(t)\}$ with rate λ with rateλ for some $\lambda > 0$.

   Does this process have a stationary distribution? If so, find it. If not, explain why it does not.

6. Suppose that d particles are distributed into two cells. A particle in cell 1 remains in that cell for a random length of time that is exponentially distributed with rate λ before moving to cell 2. A particle in cell 2 remains in that cell for a random length of time that is exponentially distributed with rate μ before moving to cell 1. The particles act independently of each other. Let $X(t)$ denote the number of paticles in cell 2 at time $t \geq 0$. The $X(t)$ is a birth-and-death process on $\{0, 1, 2, \dots, d\}$.

   (a) Find the stationary distribution for $X(t)$.

   (b) When the system is in equlibrium, find $\mathsf{E}[X(t)]$.

7. For the M/M/1 queue, find the mean queue length in equilibrium

   (a) "from scratch"

   (b) using the formula we derived for the M/G/1 queue

8. Determine the mean waiting time for a customer in an M/M/2 system when $\lambda = 2$ and $\mu = 1.2$. Compare this with the mean waiting time for a customer in an M/M/1 system with $\lambda = 1$ and $\mu = 1.2$. Why is there a difference when the arrival rate per server is the same in both cases?

9. Consider the M/G/1 queue, where the service times have a $\Gamma(2, \nu)$ distribution. Assume the system is in equilibrium.

(a) What is the probability that an incoming customer has to wait for service?

(b) What is the expected queue length in equilibrium?

10. Consider a simple M/M/1 queue with arrival rate λ and service rate μ. Find the distribution of the number of customers who arrive while a given customer is being served. Assume the queue is in equilibrium.

11. Consider an M/M/s queue in equilibrium with the normal arrival and service rate parameters λ and μ. Let L be the mean number of customers in the system and let L_0 be the mean number of customers in the system waiting for, but not undergoing, service.

    One can show (but you don't have to) that

$$L_0 = \frac{\pi_0}{s!}\left(\frac{\lambda}{\mu}\right)^s \frac{(\lambda/s\mu)}{(1-\lambda/s\mu)^2}$$

    Relate L and L_0. That is, fill in the question mark in

$$L = L_0 + ?$$

12. Describe the accept-reject algorithm, including what it's used for.

13. Describe the Metropolis-Hastings algorithm, including what it's used for.

14. Suppose we have a CTMC with generator matrix $Q = [q_{ij}]$. Let π be a distribution on the state space.

    (a) What does it mean when we say that π has "detailed balance" with respect to the generator?

    (b) If π does have deatiled balance with respect to the generator, show that π is a stationary distribution for the chain.

15. Consider a machine that is, at any given time, either in an operating state or a repair state. Suppose that

    Let

$$X(t) = \begin{cases} 0 & , \text{ the machine is operating at time } t \\ 1 & , \text{ the machine is in repair at time } t \end{cases}$$

    What is the probability that the machine is in the operating state at time $t > 0$ given that it was operating at time 0

    Hint: Solve the Kolmogorov forward equation.