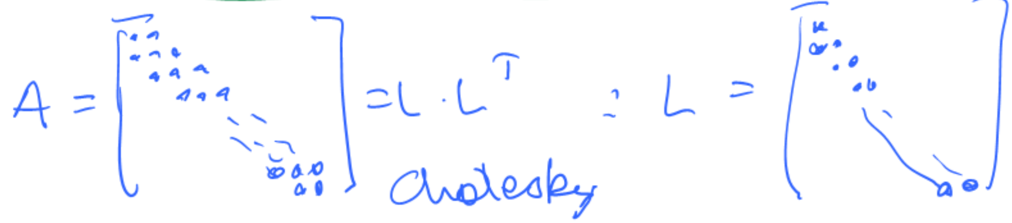
Reverse Cuthill-McKee algorithm.

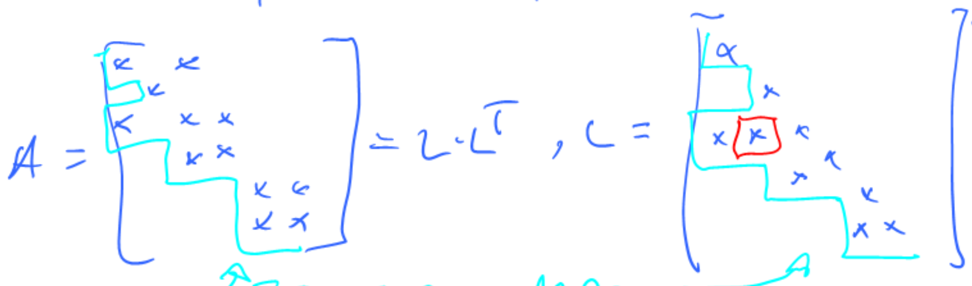
Exchange order of equations and unknowns for reduced bandwidth. Matlab: symrcm.

Ex 1:



No extra 'fill-in'. Perfect case..

Ex 2:



same envelope

Small 'fill-in'.

Here harmless. just one entry, but big issue if originally lots of zeros inside 'skyline'.

Question: Can we permute system $A\underline{x} = \underline{b}$ to $PAP^TP\underline{x} = P\underline{b}$, P permutation matrix to minimize fill-in?

Key quantities:

β_i = On row i, 'distance' from leftmost entry to diagonal.

$\beta_{max} = \max_i \beta_i$

$e = \sum_i \beta_i$.

In Example 2:

$$\left.\begin{matrix} i = & 1 & 2 & 3 & 4 & 5 & 6 \\ \beta_i = & 0 & 0 & 2 & 1 & 0 & 1 \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} \beta_{max} = 2 \\ e = 4 \end{matrix}\right\}$$

Ex 3:

$$A = \begin{bmatrix} x & x & x & x \\ x & x & & \\ x & & x & \\ x & & & x \end{bmatrix} \quad \text{Arrowhead} \left\{ \begin{array}{l} \beta_{max} = 3 \\ \varepsilon = 6 \end{array} \right.$$

Instead of solving $A\underline{x} = \underline{b}$; solve $\underbrace{PAP^T}\, P\underline{x} = P\underline{b}$

Use for ex.

$$P = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}; \text{ Then } PAP = \begin{bmatrix} x & & & x \\ & x & & x \\ & & x & x \\ x & x & x & x \end{bmatrix} \left\{ \begin{array}{l} \beta_{max} = 3 \\ \varepsilon = 3 \end{array} \right.$$

Optimal ??

We will see:
- This was not optimal
- We need a systematic process to find P.

Some illustrative examples:

Ex 4:

$$\begin{bmatrix} x & x & & & \\ x & x & x & & \\ & x & x & x & \\ & & x & x & x \\ & & & x & x \end{bmatrix}$$

Draw graph, much as for extended Gershgorin's theorem, (to check irreducibility), but now lines only; no arrows.

$1 - 2 - 3 - 4 - 5$

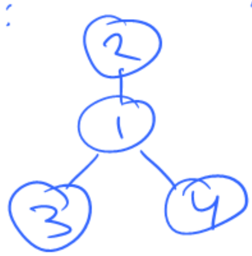
We can from graph read off

$$\left\{ \begin{array}{l} \beta_1 = \max(0, 1-2) = 0 \\ \beta_2 = \max(0, 2-1, 2-3) = 1 \\ \vdots \\ \beta_5 = \max(0, 5-4) = 1 \end{array} \right.$$

$$\Rightarrow \left\{ \begin{array}{l} \beta_{max} = 1 \\ \varepsilon = 4 \end{array} \right.$$

Ex 5: Arrowhead:

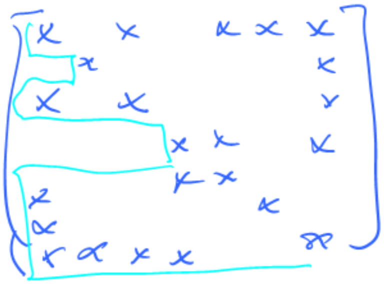
$$\begin{bmatrix} x & x & x & x \\ x & x & & \\ x & & x & \\ x & & & x \end{bmatrix}$$

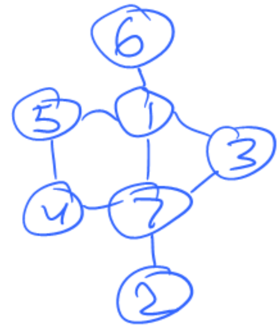


$$\left\{ \begin{array}{ll} \beta_1 = \max(0, 1-2, 1-3, 1-4) & = 0 \\ \beta_2 = \max(0, 2-1) & = 1 \\ \beta_3 = \max(0, 3-1) & = 2 \\ \beta_4 = \max(0, 4-1) & = 3 \end{array} \right.$$

So (as before) $\left\{ \begin{array}{l} \beta_{max} = 3 \\ \varepsilon = 6 \end{array} \right.$

Ex 6:





$\beta_1 = 0$
$\beta_2 = 0$
$\beta_3 = 2$
$\beta_4 = 0$
$\beta_5 = 4$
$\beta_6 = 5$
$\beta_7 = 6$

$\begin{cases} \beta_{max} = 6 \\ \Sigma = 17 \end{cases}$

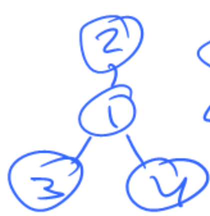
Note: Different P correspond to different orderings of the numbers 1, 2, ... within the graph.

Cuthill-McKee idea:

→ Start 'far out' in the graph (max distance to other elements)
→ Fill in 1, 2, 3 ... ensuring that neighbors to low elements get filled first.
→ Reverse.

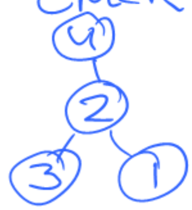
Ex 5 (re-visited) Arrowhead:

Head:



$\begin{cases} \beta_{max} = 3 \\ \Sigma = 6 \end{cases}$

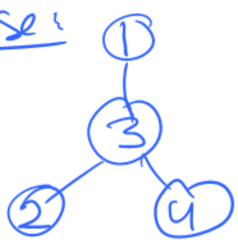
CMcK:



$\beta_1 = \max(0, 1-2) = 0$
$\beta_2 = \max(0, 2-1, 2-3, 2-4) = 1$
$\beta_3 = \max(0, 3-2) = 1$
$\beta_4 = \max(0, 4-2) = 2$

$\begin{cases} \beta_{max} = 2 \\ \Sigma = 4. \end{cases}$

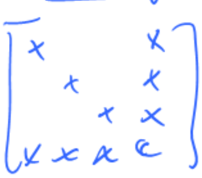
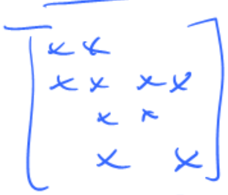
Reverse:



$\beta_1 = \max(0, 1-3) = 0$
$\beta_2 = \max(0, 2-3) = 0$
$\beta_3 = \max(0, 3-1, 3-2, 3-4) = 2$
$\beta_4 = \max(0, 4-3)\ 1$

$\begin{cases} \beta_{max} = 2 \\ \Sigma = 3 \end{cases}$

Sequence:

original	looked good	Cuthill-McKee	Reverse Cuthill-McKee
			
$\begin{cases} \beta_{max} = 3 \\ \Sigma = 6 \end{cases}$	$\begin{cases} \beta_{max} = 3 \\ \Sigma = 3 \end{cases}$	$\begin{cases} \beta_{max} = 2 \\ \Sigma = 4 \end{cases}$	$\begin{cases} \beta_{max} = 2 \\ \Sigma = 3 \end{cases}$

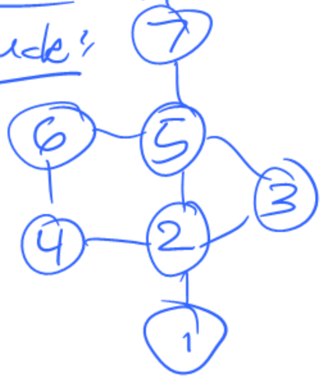
Theorem: In the reverse step:

→ β_{max} always same

→ ε may decrease (never increase)

Ex 6: (Re-visited)

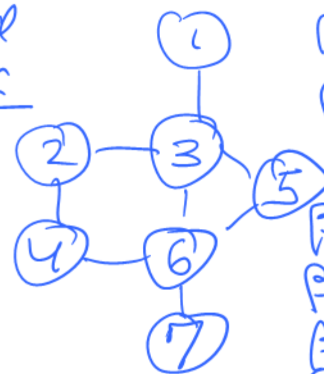
C-McK:



$\beta_1=0$
$\beta_2=1$
$\beta_3=1$
$\beta_4=2$
$\beta_5=3$
$\beta_6=2$
$\beta_7=2$

$\left\{\begin{array}{l}\beta_{max}=3\\ \varepsilon=11\end{array}\right.$

Reversed C-McK



$\beta_1=0$
$\beta_2=0$
$\beta_3=2$
$\beta_4=2$
$\beta_5=2$
$\beta_6=3$
$\beta_7=1$

$\left\{\begin{array}{l}\beta_{max}=3\\ \varepsilon=10\end{array}\right.$

Original $\left\{\begin{array}{l}\beta_{max}=6\\ \varepsilon=17\end{array}\right.$

Typical: Larger scale problems ⇒ bigger gain.

Issues:

1. Reverse Cuthill-McKee very good, but <u>not</u> theoretically optimal.
2. Discussion here limited to A symmetric. For non-symmetric, might need to pivot wrt conditioning as well
3. Cost increase for larger graphs. {Often not a big issue! symrmc very fast}