22. (a) Let $g_0^{(n)}$ be the probability that, starting from state 0, the first return to 0 happens in $n$ time steps.

We have


g_0^{(1)} = P(0 \rightarrow 0) = 1 - \alpha \\
g_0^{(2)} = P(0 \rightarrow 1 \rightarrow 0) = \alpha \cdot \beta \\
g_0^{(3)} = P(0 \rightarrow 1 \rightarrow 1 \rightarrow 0) = \alpha \cdot (1 - \beta) \cdot \beta \\
g_0^{(4)} = P(0 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 0) = \alpha \cdot (1 - \beta)^2 \cdot \beta \\
g_0^{(5)} = P(0 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 1 \rightarrow 0) = \alpha \cdot (1 - \beta)^3 \cdot \beta \\

\vdots \\
g_0^{(n)} = \alpha \cdot (1 - \beta)^{n-2} \cdot \beta

for $n \geq 2$.

(b) Let $N =$ number of steps to first return. Then

\[
E[X] = \sum_{n=1}^{\infty} n \cdot P(N = n) \\
= 1 \cdot P(N = 1) + \sum_{n=2}^{\infty} n \cdot P(N = n) \\
= 1 - \alpha + \sum_{n=2}^{\infty} n \cdot \alpha \cdot (1 - \beta)^{n-2} \cdot \beta \\
= 1 - \alpha + \alpha \sum_{n=0}^{\infty} (n + 2)(1 - \beta)^n \cdot \beta \\
= 1 - \alpha + \alpha \left[ \sum_{n=0}^{\infty} n(1 - \beta)^n \cdot \beta + 2 \sum_{n=0}^{\infty} (1 - \beta)^n \cdot \beta \right] \\
\text{expectation of } geom_0(p = \beta) + 1 \text{ because pdf} \\
= 1 - \alpha + \alpha \left[ \frac{1 - \beta}{\beta} + 2 \cdot 1 \right]
\]

24. (a) Yes, this is a stopping time.

(b) Yes, this is a stopping time.

(c) No, this is not a stopping time. We can’t tell, in general, that a chain has hit state 5 for the last time without knowing the future values of the chain.

(d) No, this is not a stopping time. Consider the event \( \{T = 3\} \). Note that

\[ \{T = 3\} = \{X_3 = 5, X_7 \neq 5\} \]

That is, in order to determine if 3 is the last time (of times in A) that we hit 5, we will need to check that we didn’t hit 5 at time 7. For $T$ to be a stopping time, we need the event $\{T = 3\}$ to be completely determined by $\{X_0, X_1, X_2, X_3\}$ and it is not because we need to look at $X_7$.

(e) Yes, this is a stopping time.
(f) No, this is not a stopping time. The event that \( \{W = n\} \) is equivalent to the event that \( \{T - 1 = n\} \) which is equivalent to the event that \( \{T = n + 1\} \) which is equivalent to the event that

\[
\{X_0 \geq 10, X_1 \geq 10, \ldots, X_n \geq 10, X_{n+1} < 10\}
\]

If \( W \) was a stopping time, to determine whether or not the event \( \{W = n\} \) occurs should only involve looking at \( X_0, X_1, \ldots, X_n \). Maybe less points of the chain but never more. Here we see that we also need to look at \( X_{n+1} \) which means that \( W \) is not a stopping time.

Just for fun, since the strong Markov property holds for stopping times, another way to show that \( W \) is not a stopping time is to show that the strong Markov property does not hold.

Consider

\[
P(X_{W+1} = j | X_W = i, \text{history}) = P(X_T = j | X_{T-1} = i, \text{history})
\]

Suppose that \( i = 1 \) and \( j = 15 \), we would want this to equal \( p_{1,15} \) but instead it will equal 0 since we know we must be at a value smaller than 10 at time \( T \).

(g) Yes, this is a stopping time. Although no justification is required for the “yes” answers here, let’s figure out why.

Since \( S \) is a stopping time, the event \( \{S = k\} \) is completely determined by the values of \( X_0, X_1, \ldots, X_k \).

Similarly, since \( T \) is a stopping time, the event \( \{T = k\} \) is completely determined by the values of \( X_0, X_1, \ldots, X_k \).

Consider the event \( \{S + T = n\} \). Note that

\[
\{S + T = n\} = \bigcup_{k=0}^{n} \{S = k, T = n - k\} = \bigcup_{k=0}^{n} \{S = k\} \cap \{T = n - k\}
\]

The event \( \{S = k, T = n - k\} \) is completely determined by looking at \( X_0, X_1, \ldots, X_k \) and \( X_0, X_1, \ldots, X_{n-k} \) which means we have to look at

\[
X_0, X_1, \ldots, X_{\max(k,n-k)}
\]

to determine whether both “sub-events” occurred.

Since \( k \) goes from 0 to \( n \), the occurrence or non-occurrence of the event \( \{S + T = n\} \) can be completely determined by \( X_0, X_1, \ldots, X_n \).

25. First, let’s solve for the \( \pi \)’s. The stationary equations are

\[
\begin{align*}
\pi_0 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\
\pi_1 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\
\pi_2 &= 0.8\pi_0 + 0.6\pi_1 + 0.4\pi_2
\end{align*}
\]

with the additional constraint that \( \pi_0 + \pi_1 + \pi_2 = 1 \).

The solution is

\[
\pi_0 = \frac{3}{13} \quad \pi_1 = \frac{3}{13} \quad \pi_2 = \frac{7}{13}
\]
On the other hand, suppose we start a chain at state 0 and want to find the expected time of first return to 0. That is, we define
\[ T_0 = \{ \min n \geq 1 : X_n = 0 \}. \]
Define \( u_i = \mathbb{E}_i[T_0] = \mathbb{E}[T_0|X_0 = i] \).

We want to find \( u_0 \) and to see that it is equal to \( \frac{13}{3} \).

By a first step analysis, we have
\[
\begin{align*}
    u_0 &= 1 + (0.1)(0) + (0.1)u_1 + (0.8)u_2 \\
    u_1 &= 1 + (0.2)(0) + (0.2)u_1 + (0.6)u_2 \\
    u_2 &= 1 + (0.3)(0) + (0.3)u_1 + (0.4)u_2
\end{align*}
\]

The solution is
\[
\begin{align*}
    u_0 &= \frac{13}{3} \\
    u_1 &= 4 \\
    u_2 &= \frac{11}{3}
\end{align*}
\]

So, we did, in fact, see \( u_0 = \frac{1}{\pi_0} \).

(We do not want this \( u_1 \) to be \( \frac{1}{\pi_1} \). We need to redefine the stopping time to stop at 1 and do the first step analysis all over again to find the mean time, starting at 1 until we return to 1. We would have to do it a third time to verify that \( \mathbb{E}_2[T_2] = \frac{1}{\pi_2} \).)