APPM 4/5560

Review Solutions Correction and Addendum

22. (a) Let $g_0^{(n)}$ be the probability that, starting from state 0, the <u>first</u> return to 0 happens in n time steps.

We have

$$
\begin{aligned}
g_0^{(1)} &= P(0 \to 0) = 1-\alpha \\
g_0^{(2)} &= P(0 \to 1 \to 0) = \alpha \cdot \beta \\
g_0^{(3)} &= P(0 \to 1 \to 1 \to 0) = \alpha \cdot (1-\beta) \cdot \beta \\
g_0^{(4)} &= P(0 \to 1 \to 1 \to 1 \to 0) = \alpha \cdot (1-\beta)^2 \cdot \beta \\
g_0^{(5)} &= P(0 \to 1 \to 1 \to 1 \to 1 \to 0) = \alpha \cdot (1-\beta)^3 \cdot \beta \\
&\vdots \\
g_0^{(n)} &= \alpha \cdot (1-\beta)^{n-2} \cdot \beta
\end{aligned}
$$

for $n \geq 2$.

(b) Let $N =$ number of steps to first return. Then

$$
\begin{aligned}
\mathsf{E}[X] &= \sum_{n=1}^{\infty} n \cdot P(N=n) \\
&= 1 \cdot P(N=1) + \sum_{n=2}^{\infty} n \cdot P(N=n) \\
&= 1-\alpha + \sum_{n=2}^{\infty} n \cdot \alpha \cdot (1-\beta)^{n-2} \cdot \beta \\
&= 1-\alpha + \alpha \sum_{n=0}^{\infty} (n+2)(1-\beta)^n \cdot \beta \\
&= 1-\alpha + \alpha[\underbrace{\sum_{n=0}^{\infty} n(1-\beta)^n \cdot \beta}_{\substack{\text{expectation} \\ \text{of } geom_0(p=\beta)}} + 2\underbrace{\sum_{n=0}^{\infty} (1-\beta)^n \cdot \beta]}_{\substack{\text{1 because} \\ \text{pdf}}} \\
&= 1-\alpha+\alpha\left[\frac{1-\beta}{\beta} + 2 \cdot 1\right]
\end{aligned}
$$

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24. (a) Yes, this is a stopping time.

(b) Yes, this is a stopping time.

(c) No, this is not a stopping time. We can't tell, in general, that a chain has hit state 5 for the last time without knowing the future values of the chain.

(d) No, this is not a stopping time. Consider the event $\{T=3\}$. Note that

$$\{T=3\} = \{X_3 = 5, X_7 \neq 5\}$$

That is, in order to determine if 3 is the last time (of times in A) that we hit 5, we will need to check that we didn't hit 5 at time 7. For T to be a stopping time, we need the event $\{T=3\}$ to be completely determined by $\{X_0, X_1, X_2, X_3\}$ and it is not because we need to look at X_7.

(e) Yes, this is a stopping time.

(f) No, this is not a stopping time.

The event that $\{W = n\}$ is equivalent to the event that $\{T - 1 = n\}$ which is equivalent to the event that $\{T = n + 1\}$ which is equivalent to the event that

$$\{X_0 \geq 10, X_1 \geq 10, \ldots, X_n \geq 10, X_{n+1} < 10\}$$

If W was a stopping time, to determine whether or not the event $\{W = n\}$ occurs should only involve looking at $X_0, X_1, \ldots, X_n$. Maybe less points of the chain but never more. Here we see that we also need to look at X_{n+1} which means that W is not a stopping time.

Just for fun, since the strong Markov property holds for stopping times, another way to show that W is not a stopping time is to show that the strong Markov property does not hold.

COnsider

$$P(X_{W+1} = j | X_W = i, \text{history}) = P(X_T = j | X_{T-1} = i, \text{history})$$

Suppose that $i = 1$ and $j = 15$, we would want this to equal $p_{1,15}$ but instead it will equal 0 since we know we must be at a value smaller than 10 at time T.

(g) Yes, this is a stopping time. Although no justification is required for the "yes" answers here, let's figure out why.

Since S is a stopping time, the event $\{S = k\}$ is completely determined by the values of $X_0, X_1, \ldots, X_k$.

Similarly, since T is a stopping time, the event $\{T = k\}$ is completely determined by the values of $X_0, X_1, \ldots, X_k$.

Consider the event $\{S + T = n\}$. Note that

$$\{S + T = n\} = \bigcup_{k=0}^{n} \{S = k, T = n - k\} = \bigcup_{k=0}^{n} [\{S = k\} \cap \{T = n - k\}]$$

The event $\{S = k, T = n - k\}$ is completely determined by looking at $X_0, X_1, \ldots, X_k$ and $X_0, X_1, \ldots, X_{n-k}$ which means we have to look at

$$X_0, X_1, \ldots, X_{\max(k, n-k)}$$

to determine whether both "sub-events" occurred.

Since k goes from 0 to n, the occurrence or non-occurrence of the event $\{S + T = n\}$ can be completely determined by $X_0, X_1, \ldots, X_n$.

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25. First, let's solve for the π's. The stationary equations are

$$\begin{aligned}
\pi_0 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\
\pi_1 &= 0.1\pi_0 + 0.2\pi_1 + 0.3\pi_2 \\
\pi_2 &= 0.8\pi_0 + 0.6\pi_1 + 0.4\pi_2
\end{aligned}$$

with the additional constraint that $\pi_0 + \pi_1 + \pi_2 = 1$.

The solution is

$$\pi_0 = \frac{3}{13} \qquad \pi_1 = \frac{3}{13} \qquad \pi_2 = \frac{7}{13}.$$

On the other hand, suppose we start a chain at state 0 and want to find the expected time of first return to 0. That is, we define

$$T_0 = \{\min n \geq 1 : X_n = 0\}.$$

Define $u_i = \mathsf{E}_i[T_0] = \mathsf{E}[T_0|X_0 = i]$.

We want to find u_0 and to see that it is equal to 13/3.

By a first step analysis, we have

$$u_0 = 1 + (0.1)(0) + (0.1)u_1 + (0.8)u_2$$

$$u_1 = 1 + (0.2)(0) + (0.2)u_1 + (0.6)u_2$$

$$u_2 = 1 + (0.3)(0) + (0.3)u_1 + (0.4)u_2$$

The solution is

$$u_0 = \frac{13}{3} \qquad u_1 = 4 \qquad u_2 = \frac{11}{3}$$

So, we did, in fact, see $u_0 = 1/\pi_0$.

(We do not want this u_1 to be $1/\pi_1$. We need to redefine the stopping time to stop at 1 and do the first step analysis all over again to find the mean time, starting at 1 until we return to 1. We would have to do it a third time to verify that $\mathsf{E}_2[T_2] = 1/\pi_2$.)