APPM 4/5560

Solutions to Review Problems for Exam One

1. (a)

$$\begin{aligned} P(\text{2nd is A}) &= P(\text{2nd is A}|\text{1st is A}) \cdot P(\text{1st is A}) + P(\text{2nd is A}|\text{1st is not A}) \cdot P(\text{1st is not A}) \\ &= \tfrac{3}{51} \cdot \tfrac{4}{52} + \tfrac{4}{51} \cdot \tfrac{48}{52} \end{aligned}$$

(b)

$$\begin{aligned} P(\text{1st was A}|\text{2nd is A}) &= \frac{P(\text{2nd is A} \mid \text{1st was A}) \cdot P(\text{1st was A})}{P(\text{2nd is A})} \\ &= \frac{\frac{3}{51} \cdot \frac{4}{52}}{\frac{3}{51} \cdot \frac{4}{52} + \frac{4}{51} \cdot \frac{48}{52}} \end{aligned}$$

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2. Imagine the cards being laid down in "slots". For any particular slot considered alone, you have a 1 in 13 chance of putting down the right card. So,

$$\mathsf{E}[I_i] = P(I_i = 1) = P(\,ith \text{ card dealt is a match}\,) = \frac{1}{13}$$

So,

$$\mathsf{E}[N] = \mathsf{E}[\sum_{i=1}^{13} I_i] = \sum_{i=1}^{13} \mathsf{E}[I_i] = \sum_{i=1}^{13} \frac{1}{13} = 1$$

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3. (a)

$$\begin{aligned} P(X_1 = 0) &= P(X_1 = 0|X_0 = 0) \cdot P(X_0 = 0) + P(X_1 = 0|X_0 = 1) \cdot P(X_0 = 1) \\ &= (0.8)(0.5) + (0.5)(0.5) = 0.65 \end{aligned}$$

(b)

$$P(X_3 = 1|X_0 = 0) = \text{ the "0-1 entry" of } \mathbf{P}^{(3)} = 0.278$$

(c) The answer is π_0, where π_0 is the solution to the following system of equations.

$$\begin{array}{ccccc} \pi_0 & = & 0.8\pi_0 & + & 0.5\pi_1 \\ \pi_1 & = & 0.2\pi_0 & + & 0.5\pi_1 \\ \pi_0 & + & \pi_1 & = & 1 \end{array}$$

The answer is $\pi_0 = 5/7 \approx 0.7143$.

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4. The communication classes are $\{0, 1, 2\}$, $\{3\}$, and $\{4\}$. Since this is not one big set, the chain is reducible. (So, no, it is not irreducible.) As for the classification of states:

- State 4 is very clearly recurrent, since, starting at 4, the chain will definitely return to 4 with probability 1. (In fact, it will stay at 4 forever!)
- State 3 is transient because there is a positive probability you will leave this state and never return. Specifically, with probability 1/3, you will leave state 3 forever.
- States 0, 1, and 2 are recurrent since they all communicate and there is no leaving this class. Sometimes it is difficult to envision that we will return to a state with probability one– instead, look at this the opposite way: from each of these states, is there some positive probability that you can move to a place from where you can't return? In other words,

  $$P_i(\text{ will return to } i \text{ }) < 1$$

  happens if and only if

  $$P_i(\text{ won't return to } i \text{ }) > 0.$$

  happens if and only if

  If there is a way out of the class $\mathcal{C}_1 := \{0, 1, 2\}$, then the states in there would be transient because have found a positive probability of not returning. ie: If, for $i \in \{0, 1, 2\}$ and $j \notin \{0, 1, 2\}$, we had $i \to j$, then there will be no way back to i. If there was a way back, then i and j would communicate and so j should have been in $\mathcal{C}_1$ to begin with.

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5. Let X_n be the number of red balls in the urn at time step n. (We could have instead defined this to be the number of yellow balls in the urn at time n.)

   The state space is $\{0, 1, 2\}$ and 0 is an absorbing state since, once there are zero red balls, there will always be zero red balls. (We are only replacing balls with yellow ones.)

   The transition probability matrix is

   $$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} \end{array} \right\| \end{array}$$

   Let $T = \min\{n \geq 0 : X_n = 0\}$ be the duration of the process. Let $u_i = \mathsf{E}[T]|X_0 = i]$. Then, we want u_2.

   - Since, starting with two red balls, we have to go at least one time step to move this Markov chain towards zero, we write

     $$u_2 = 1 + \cdots$$

   - With probability 2/3, we will have moved to state 1 and then we must start counting the expected number of time steps to reach state 0 starting from state 1. This is precisely u_1. So

     $$u_2 = 1 + \frac{2}{3}u_1 + \cdots$$

- With probability 1/3, we will have moved to (stayed at) state 2 and then we must start counting the expected number of time steps to reach state 0 starting from state 2. This is precisely u_2. So

$$u_2 = 1 + \frac{2}{3}u_1 + \frac{1}{3}u_2$$

Preceeding in the same manner, we get

$$u_1 = 1 + \frac{1}{3}u_0 + \frac{2}{3}u_1$$

and

$$u_0 = 0 \qquad \text{by definition.}$$

The answer is then $u_2 = 9/2$.

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6. Let X_n be the compartment the rat is in at time n. The state space is $\{1, 2, 3, 4, 5, 6\}$. As for the transition probability matrix, should we make "shock" and "food" absorbing states, or do we assume that the rat keeps randomly wandering even after he, for example, finds food? It doesn't matter, you will arrive at the same answer since we are stopping the chain at shock or food and we don't need to consider transitions from there. In these solutions, I will not make them absorbing because, technically, it doesn't say that the rat stops moving when he finds food (or a shock!).

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left\| \begin{array}{cccccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right\| \end{array}$$

(a) Define $T = \min\{n \geq 0 : X_n = 1 \text{ or } X_n = 6\}$. Also, define $p_i = P(X_T = 6 | X_0 = i)$. Then we want to find p_4.

- Starting at 4, we will move to 3 with probability 1/3, and then, since transitions are independent in a Markov chain, we will have to multiply by the probability that we are 6 at time T given that we started at state 3.

$$p_4 = \frac{1}{3}p_3 + \cdots$$

- Starting at 4, we will move to 5 with probability 1/3, and then, since transitions are independent in a Markov chain, we will have to multiply by the probability that we are 6 at time T given that we started at state 5.

$$p_4 = \frac{1}{3}p_3 + \frac{1}{3}p_5 + \cdots$$

- Finally, starting at 4, we will move to 6 with probability 1/3, and then, since transitions are independent in a Markov chain, we will have to multiply by the probability that we are 6 at time T given that we started at state 6. This probability is $p_6 = 1$

$$p_4 = \frac{1}{3}p_3 + \frac{1}{3}p_5 + \left(\frac{1}{3}\right)(1)$$

We get the remaining equations in the system in a similar fashion:

$$p_1 = 0$$

$$p_2 = \left(\frac{1}{2}\right)(0) + \frac{1}{2}p_3$$

$$p_3 = \frac{1}{2}p_2 + \frac{1}{2}p_4$$

$$p_5 = 1 \cdot p_4$$

Thus, we have a system to solve!

(Note: The p_i need not add up to one. They are "different universe" probabilities.)

(b) The way I have set this problem up, all states communicate. (The rat can walk from anywhere to anywhere else.) If you considered 1 and 6 to be absorbing, then all states do not communicate and your answer will be different.

In my system, every state has period 2. (Since the state space is irreducible, you only have to consider the periodicity of a single state.) This is like a random walk: when you are at 1, for example, you will definitely leave 1 but you could come right back in two time steps: $1 \to 2 \to 1$, or four: $1 \to 2 \to 3 \to 2 \to 1$, etc... The greatest common divisor of all time steps to return is 2.

All states are periodic with period 2!

(If you set up the transition matrix so that states 1 and 6 are absorbing, then states 1 and 6 have period 1 (aperiodic states) while the rest of the states have period 2 (periodic with period 2). The entire chain is said to be (a)periodic if all states are. Since we have two kinds of states, the chain is <u>neither</u> periodic nor aperiodic.)

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7. (a) The answer is π_1 which is obtained by solving the system:

$$\begin{array}{ccccccccc} \pi_0 & = & \frac{1}{3}\pi_0 & & & + & \frac{3}{4}\pi_2 & & \\ \pi_1 & = & \frac{1}{3}\pi_0 & + & \frac{1}{2}\pi_1 & + & \frac{1}{4}\pi\ 2 & & \\ \pi_2 & = & \frac{1}{3}\pi_0 & + & \frac{1}{2}\pi_1 & & & & \\ \pi_0 & + & \pi_1 & + & \pi_2 & + & \pi_3 & = & 1 \end{array}$$

The solution is

$$\pi_0 = 1/3, \qquad \boxed{\pi_1 = 10/27}, \qquad \pi_2 = 8/27$$

(b) Let $T = \min\{n \geq 0 : X_n = 1\}$ and define $u_i = \mathsf{E}[T|X_0 = i\ ]$. Then we want u_0.

- Since we must take at least one step in our path towards 1 when we start from 0, we have that
$$u_0 = 1 + \cdots$$
- With probability 1/3, we will be back at 0 and have to start counting again from there:
$$u_0 = 1 + \frac{1}{3}u_0 + \cdots$$
- With probability 1/3 we will move from 0 to 1 and stop counting. ie: we count zero more steps:
$$u_0 = 1 + \frac{1}{3}u_0 + + \frac{1}{3}(0) + \cdots$$
- With probability 1/3, we will be at state 2 and have to start counting from there:
$$u_0 = 1 + \frac{1}{3}u_0 + + \frac{1}{3}(0) + \frac{1}{3}u_2$$

Similarly, we get that
$$u_1 = 0$$
and
$$u_2 = 1 + \frac{3}{4}u_0 + \frac{1}{4}(0).$$
Solving this system gives
$$\boxed{u_0 = 16/5}, \qquad u_2 = 17/5$$

(c) Let $T = \min\{n \geq 0 : X_n = 1 \text{ or } X_n = 2\}$ and define $p_i = P(X_T = 2 | X_0 = i)$. Then we want p_0.

- Starting from 0, with probability 1/3 we stay at zero and have to now compute the desired probability from the beginning:
$$p_0 = \frac{1}{3}p_0 + \cdots$$
- With probability 1/3, we move to state 1. Then we will have probabilty zero of ending our path in state 2 since we travel only until we hit state 1 or state 2.
$$p_0 = \frac{1}{3}p_0 + \frac{1}{3}(0) + \cdots$$
- With probability 1/3, we move to state 2. Then we will have probabilty one of ending our path in state 2 since we travel only until we hit state 1 or state 2.
$$p_0 = \frac{1}{3}p_0 + \frac{1}{3}(0) + \frac{1}{3}(1)$$

This is the only equation we need to solve for p_0:
$$p_0 = \frac{1}{2}.$$

(d) Let $T = \min\{n \geq 0 : X_n = 2\}$ and define w_{ij} to be the mean amount of time the process spends in state j when started in i and ended at 2. (Actually, we really don't need to define T here.)

Then we want w_{01}.

- Starting at state 0, with probability 1/3, we move to state 0 and then we have count the mean number of times the process spends in state 1 before hitting state 2, starting all over again from state 0:

$$w_{01} = \frac{1}{3}w_{01} + \cdots$$

- With probability 1/3, we move to state 1 and now we have to start counting the mean number of times a process, starting in state 1, visits state 1 before hitting state 2:

$$w_{01} = \frac{1}{3}w_{01} + \frac{1}{3}w_{11} + \cdots$$

- With probability 1/3, we will go straight from state 0 to state 2, at which time we stop observing the process without having counted any ones:

$$w_{01} = \frac{1}{3}w_{01} + \frac{1}{3}w_{11} + \frac{1}{3}(0)$$

Similarly, we give the expression for w_{11}:

$$w_{11} = 1 + \frac{1}{2}w_{11} + \frac{1}{2}(0)$$

(We started with "one plus", because, starting in state 1 and observing a process that ends in state 2, we count at least one 1.)
Solving this system gives

$$\boxed{w_{01} = 1}, \qquad w_{11} = 2.$$

(e) This one is very similar to part (d) above. We just count things a little differently because we are looking at a cycle from state 0 to state 0 which includes a first and second hitting of zero as opposed to simply a first hitting of zero.
If we let w_{ij} be the mean time the process spends in state j when started from i before returning to zero, we get the system

$$w_{01} = \frac{1}{3}(0) + \frac{1}{3}w_{11} + \frac{1}{3}w_{21}$$

$$w_{11} = 1 + \frac{1}{2}w_{11} + \frac{1}{2}w_{21}$$

and

$$w_{21} = \frac{3}{4}(0) + \frac{1}{4}w_{11}$$

which has solution

$$w_{01} = 10/9, \qquad w_{11} = 8/3, \qquad w_{21} = 2/3.$$

So, the answer is 10/9.

(f)

$$P(X_2 = 1 | X_2 \neq 2, X_1 \neq 2, X_0 = 0) = \frac{P(X_2 = 1, X_2 \neq 2,\ X_1 \neq 2 | X_0 = 0)}{P(X_2 \neq 2, X_1 \neq 2 | X_0 = 0)}$$

$$= \frac{P(X_2 = 1, X_1 \neq 2 | X_0 = 0)}{P(X_2 \neq 2, X_1 \neq 2 | X_0 = 0)}$$

$$= \frac{q_{01}^{(2)}}{1 - q_{02}^{(2)}}$$

where $q_{ij}^{(2)}$ is the $i-j$th entry of the matrix $Q^{(2)} = Q^2$ where

$$\mathbf{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{array} \right\| \end{array}$$

$$\frac{q_{01}^{(2)}}{1 - q_{02}^{(2)}} = \frac{5/18}{1 - 11/18} = \boxed{\tfrac{5}{7}}.$$

If we already know that $X_2 \neq 2$, then it can only be 0 or 1, so the probability that it is 1 given that it is not 2 is higher than the probability that it is 1 given that it could be any of 0, 1, or 2. Therefore,

$$P(X_2 = 1 | X_1 \neq 2, X_0 = 0) \leq P(X_2 = 1 | X_2 \neq 2, X_1 \neq 2, X_0 = 0).$$

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8. Let $\{X_n\}$ be a Markov chain on $\{0, 1, 2\}$ with transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right\| \end{array}$$

Then every state is periodic with period 3.

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9. (a) Using "R" to denote "rain" and "N" to denote "no rain", the four states are RR, RN, NR, and NN. Since the weather today depends only on the weather on the last two days and since two days has been lumped into one state, the system is a Markov chain. The transition probability matrix is

$$\mathbf{P} = \begin{array}{c} \\ RR \\ RN \\ NR \\ NN \end{array} \begin{array}{c} \begin{array}{cccc} RR & RN & NR & NN \end{array} \\ \left\| \begin{array}{cccc} 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 \\ 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{array} \right\| \end{array}$$

(b) The pattern is $N, N, \_\_, R$. ie: it is N, N, R, R or N, N, N, R which can be thought of as

$$NN \to NR \to RR \qquad \text{or} \qquad NN \to NN \to NR$$

but which also can be thought of as going from NN to RR or from NN to NR in two steps. I will take the second approach.... after squaring the matrix $\mathbf{P}$, the NN to RR entry is 0.18 and the NN to NR entry is 0.21.
So, the final answer is

$$0.18 + 0.21 = 0.39$$

(c) Let the states RR, RN, NR, and NN be denoted by $0, 1, 2$, and 3, respectively. Then the stationary distribution is given by the system

$$\begin{array}{rcllll} \pi_0 &=& 0.6\pi_0 & & + \ 0.6\pi_2 & \\ \pi_1 &=& 0.4\pi_0 & & + \ 0.4\pi_2 & \\ \pi_2 &=& & 0.6\pi_1 & + & 0.3\pi_3 \\ \pi_3 &=& & 0.4\pi_1 & + & 0.7\pi_3 \end{array}$$

along with the condition $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$.

The solution is then

$$\pi_0 = 9/29, \qquad \pi_1 = 6/29, \qquad \pi_2 = 6/29, \qquad \pi_3 = 8/29.$$

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10. If $i \to j$, there exists some integer $n \geq 0$ such that $p_{ij}^{(n)} > 0$. By definition of stationarity,

$$\pi_j = \sum_{k \in S} \pi_k \, p_{kj}^{(n)}$$

which is

$$\geq \pi_i \, p_{ij}^{(n)} > 0$$

by assumption.

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11. (a)

$$\begin{aligned} P(X_2 = 2) &= P(X_2 = 2 | X_0 = 1) \cdot P(X_0 = 1) + P(X_2 = 2 | X_0 = 2) \cdot P(X_0 = 2) \\ &= \tfrac{1}{2} p_{12}^{(2)} + \tfrac{1}{2} p_{22}^{(2)} \\ &= \tfrac{1}{2}(0.52) + \tfrac{1}{2}(0.61) = 0.565 \end{aligned}$$

(b) We want π_2.

$$\begin{array}{rcll} \pi_1 &=& 0.6\pi_1 & + \ 0.3\pi_2 \\ \pi_2 &=& 0.4\pi_1 & + \ 0.7\pi_2 \end{array}$$

Also, $\pi_1 + \pi_2 = 1$.

The solution is $\pi_2 \approx 0.571429$.

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12. Let X_n = the number of red bugs in the urn. Then $\{X_n\}$ is a Markov chain on $\mathcal{X} = \{0, 1, 2\}$.

The probability transition matrix is

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 \end{array} \right\| \end{array}$$

Let $T = \min\{n \geq 0 : X_n = 0\}$. Define $u_i = \mathsf{E}[T|X_0 = i]$. Then we want u_2.

$$u_2 = 1 + \frac{2}{3}u_1 + \frac{1}{3}u_2$$
$$u_1 = 1 + \frac{1}{2}(0) + \frac{1}{2}u_1$$
$$\Rightarrow u_1 = 2$$
$$\Rightarrow u_2 = 7/2.$$

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13. (a) Let $T = \min\{n \geq 0 : X_n = 0 \text{ or } X_n = 2\}$.
Let $p_i = P(X_T = 0|X_0 = i)$. Then we want p_1.

$$p_1 = (0.2)(1) + 0.7p_1 + (0.1)(0)$$

(Yes, it said "do not solve". However, $p_1 = 2/3$.)

(b) Let $u_i = \mathsf{E}[T|X_0 = i]$. Then we want u_1.

$$u_1 = 1 + (0.2)(0) + 0.7u_1 + (0.1)(0)$$

(Okay, we can solve this one too! $u_1 = 10/3$.)

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14. The communication classes are

$$\{0\}, \qquad \{1,2\}, \qquad \{3,4\}, \qquad \{5\}$$

- State 0 is recurrent.
- States 1 and 2 are recurrent.
- States 3 and 4 are transient.
- State 5 is recurrent.

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15. Define

$$\mathbf{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

Now,

$$\begin{aligned} & P(X_5 = 2|X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0) \\ = & \frac{P(X_5 = 2, X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1|X_0 = 0)}{P(X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1|X_0 = 0)} \\ = & \frac{q_{02}^{(5)}}{1 - q_{01}^{(4)}} \end{aligned}$$

where $q_{ij}^{(n)}$ is the $i-j$th entry of the $Q^{(n)} = Q^n$.

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16. (a)

$$\begin{array}{rcrcrcr} \pi_0 & = & \frac{1}{3}\pi_0 & & & + & \frac{3}{4}\pi_2 \\ \pi_1 & = & \frac{1}{3}\pi_0 & + & \frac{1}{2}\pi_1 & + & \frac{1}{4}\pi_2 \\ \pi_2 & = & \frac{1}{3}\pi_0 & + & \frac{1}{2}\pi_1 & & \end{array}$$

... and let's not forget $\pi_0 + \pi_1 + \pi_2 = 1$!

(b) Let $T = \min\{n \geq 0 : X_n = 0\}$ and define w_{ij} = the expected number of times the chain visits state j when started in state i, before time T. Then we want w_{21}. We need to solve

$$w_{21} = \frac{3}{4}(0) + \frac{1}{4}w_{11}$$

$$w_{11} = 1 + \frac{1}{2}w_{11} + \frac{1}{2}w_{21}.$$

for w_{21}.

17. The Markov chain on states S and R has probability transition matrix

$$\mathbf{P} = \begin{array}{c} \\ S \\ R \end{array} \begin{array}{cc} S & R \\ \left\| \begin{array}{cc} 0.9 & 0.1 \\ 0.8 & 0.2 \end{array} \right\| \end{array}$$

The answer is π_S, where π_S is the solution to the following system of equations.

$$\begin{array}{rcrcr} \pi_S & = & 0.9\pi_S & + & 0.8\pi_R \\ \pi_R & = & 0.1\pi_S & + & 0.2\pi_R \\ \pi_S & + & \pi_R & = & 1 \end{array}$$

The answer is $\pi_S = 8/9$.

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18. The communication classes are $\{0\}$, $\{1, 2\}$, $\{3, 4\}$, and $\{5\}$. Since we did not get one big communication class, the chain is not irreducible.

As for recurrence and transience,

State 0: State 0 is recurrent. From state 0 we will always return to state 0 in one step with probability 1.

States 1 and 2: States 1 and 2 are recurrent. There is no escaping these states!

States 3 and 4: States 3 and 4 are transient. From this class it is possible to get to state 0 after which there is no coming back!

State 5: State 5 is recurrent. From state 5 we will always return to state 5 in one step with probability 1.

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19. (a)

$$\mathbf{P} = \begin{array}{c} \\ R \\ G \\ Y \\ B \end{array} \begin{array}{c} \begin{array}{cccc} R & G & Y & B \end{array} \\ \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/7 & 1/7 & 4/7 & 1/7 \\ 1/7 & 2/7 & 4/7 & 0 \end{array} \right\| \end{array}$$

(b) Let $T = \min\{n \geq 0 : X_n = R \text{ or } X_n = G\}$. We want to find $P(X_T = R|X_0 = Y)$. For $i = R, G, Y, B$, let $p_i = P(X_T = R|X_0 = i)$. Then we want p_Y.

$$p_Y = \frac{1}{7}(1) + \frac{1}{7}(0) + \frac{4}{7}p_Y + \frac{1}{7}p_B$$

(Note that I have used that $p_R = 1$ and that $p_G = 0$.)

In order to find p_Y, we still need p_B:

$$p_B = \frac{1}{7}(1) + \frac{2}{7}(0) + \frac{4}{7}p_Y.$$

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20. Create a new Markov chain with 1 as an absorbing state. It will have transition matrix

$$\mathbf{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ \frac{3}{4} & \frac{1}{4} & 0 \end{array} \right\| \end{array}$$

The new chain acts exactly like the old chain until state 1 is hit. At this point the new chain "sticks" in state 1.

Now

$$P(X_5 = 2|X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0)$$

$$= \frac{P(X_5 = 2, X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1|X_0 = 0)}{P(X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1|X_0 = 0)}$$

The numerator is the probability, starting from state 0, that the original chain goes to state 2 in 5 time steps without ever going through state 1. This is the same as the probability that the "Q chain" goes from state 0 to state 2 in 5 time steps. This is $q_{02}^{(5)}$.

The denominator is the probability that, starting from state 0, the original chain does not hit state 1 in 4 time steps. This is 1 minus the probability that the original chain, starting in state 0, hits state 1 at least 1 time in 4 time steps. If the original chain hits state 1 in 4 time steps then the "Q chain" will be stuck in state 1 at 4 time steps. Therefore, the denominator is $1 - q_{01}^{(4)}$.

In summary,

$$P(X_5 = 2|X_4 \neq 1, X_3 \neq 1, X_2 \neq 1, X_1 \neq 1, X_0 = 0) = \frac{[Q^{(5)}]_{02}}{1 - [Q^{(4)}]_{01}}.$$

21.

$$\mathbf{P} = \begin{array}{c} \\ S \\ A \\ W \end{array} \begin{array}{c} \begin{array}{ccc} S & A & W \end{array} \\ \left\| \begin{array}{ccc} 1/2 & 1/2 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1/3 & 2/3 \end{array} \right\| \end{array}$$

Let $T = \min\{n \geq 0 : X_n = S\}$.

We want to find $\mathsf{E}[T|X_0 = W]$.

Let $u_i = \mathsf{E}[T|X_0 = i]$. Then we want to find u_W.

Now,

$$u_W = 1 + \frac{1}{3}u_A + \frac{2}{3}u_W$$

and

$$u_A = 1 + \frac{1}{4}(0) + \frac{1}{2}u_A + \frac{1}{4}u_W.$$

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22. (a) $g_0^{(n)}$ is the probability that, starting from 0 the first return to 0 happens in exactly n time steps

So,

$$f_0^{(1)} = P(0 \to 0) = 1 - \alpha$$

$$f_0^{(2)} = P(0 \to 1 \to 0) = \alpha \cdot \beta$$

$$f_0^{(3)} = P(0 \to 1 \to 1 \to 0) = \alpha \cdot (1-\beta) \cdot \beta$$

$$f_0^{(4)} = P(0 \to 1 \to 1 \to 1 \to 0) = \alpha \cdot (1-\beta)^2 \cdot \beta$$

$$f_0^{(5)} = P(0 \to 1 \to 1 \to 1 \to 1 \to 0) = \alpha \cdot (1-\beta)^3 \cdot \beta$$

$$\vdots$$

$$f_0^{(n)} = \alpha \cdot (1-\beta)^{n-2} \cdot \beta$$

for $n \geq 2$.

(b) Let $N =$ number of steps to return. Then

$$\begin{aligned} \mathsf{E}[N] &= \textstyle\sum_{n=1}^{\infty} n \cdot P(N = n) \\ &= \textstyle\sum_{n=1}^{\infty} n \cdot \alpha \cdot (1-\beta)^{n-2} \cdot \beta \end{aligned}$$

Now sum this up any way you can. The easiest way is to note that it can be written as

$$\frac{\alpha}{1-\beta} \sum_{n=1}^{\infty} n \cdot (1-\beta)^{n-1} \cdot \beta$$

and to note that the sum is the expected value of a $geom_1(p)$ random variable where $p = \beta$. This mean is $1/p = 1/\beta$.

So, the final answer is

$$\frac{\alpha}{\beta(1-\beta)}.$$

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23. One version is

$$P_{ij}^{n} = \sum_{k} P_{ik}^{(m)} \cdot P_{kj}^{(n-m)}$$

where n and m are non-negative integers with $m \leq n$.

That is, the probability that we go from state i to state j in n steps can be written as the probability of going from i to some intermediate state k in the first m steps and then from k to j in the remaining $n - m$ steps, summed over all possible k in the state space.

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24. (a) Yes, this is a stopping time.
    (b) Yes, this is a stopping time.
    (c) No, this is not a stopping time. We can't tell, in general, that a chain has hit state 5 for the last time without knowing the future values of the chain.
    (d) No, this is not a stopping time. Suppose, for example, that $T = 3$. In order to determine whether or not this is true we need to observe whether or not $X_3 = 5$. In the case that $X_3 = 5$, we will need to also observe that $X_7 \neq 5$. Thus, the occurrence event $\{T = 3\}$ can not be completely determined by looking at $\{X_0, X_1, X_2, X_3\}$ only.