Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 2024

Instructions:

Do two of three problems in each section (Prob and Stat).
Place an **X** on the lines next to the problem numbers
that you are **NOT** submitting for grading.

Prob
1. ____
2. ____
3. ____

Do not write your name anywhere on this exam.
You will be identified only by your student number.
Write this number **on each page** submitted for grading.
Show all relevant work!

Stat
4. ____
5. ____
6. ____
Total ____

Student Number ___________________________

Probability Section

Problem 1.

Let $\lambda > 0$ and $0 < p < 1$ be real constants. Suppose that $N \sim \text{Poisson}(\lambda)$ and that, conditioned on $N = n$, $B \sim \text{Binomial}(n, p)$. In particular, $B = 0$ when $N = 0$.

(a) Show that $B \sim \text{Poisson}(\lambda p)$.

(b) Without further calculations, what should the distribution of $(N - B)$ be?

(c) Finally, show that B and $(N - B)$ are independent.

Problem 2.

In what follows, $\xrightarrow{p}$ and $\xrightarrow{d}$ denote convergence in probability and distribution of random variables, respectively. Additionally, "a.s." stands for *almost surely.*

(a) Is it TRUE that if $X_n \xrightarrow{d} 0$ then $X_n \xrightarrow{p} 0$? If so, show this using the definitions of convergence in probability and distribution, otherwise provide a counter-example.

(b) Let $(X_i)_{i \geq 0}$ be a sequence of independent and identically distributed (i.i.d.) random variables with mean 2 and variance 1. Invoking well-known a.s. convergence results, which you must name explicitly as part of your solution, justify the existence of the following limit in the a.s. sense, and determine it explicitly.

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i^2}{\sqrt{n \cdot \sum_{i=1}^{n} (X_i - \bar{X})^2}}, \text{ where } \bar{X} := \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Problem 3.

In a crewless interstellar mission to an exoplanet suspected to be suitable for human life, an Artificial Intelligence (AI) is responsible for monitoring and maintaining various devices on the spacecraft. In particular, the AI immediately begins a repair whenever a device malfunctions while managing concurrently any other malfunctions. If each of the $N \geq 1$ devices on the spacecraft fails at a rate $\lambda > 0$, and the AI takes an exponentially distributed amount of time with rate $1/i$ to move from $i \geq 1$ failures to $(i-1)$ failures, address the following:

(a) Formulate a suitable Markovian model for the total number of malfunctioning devices on the spacecraft.

(b) Justify the existence of a stationary distribution for this model and determine it explicitly.

(c) Approximately what fraction of the time it will take to complete the mission will the AI be repairing at most one dysfunctional device?

Statistics Section

Problem 4.

Suppose $X_1, \ldots, X_n$ with $n > 3$ are i.i.d. exponential with rate parameter $\lambda > 0$ (that is, with mean $1/\lambda$). We will consider estimation of λ.

(a) Find the expectation of $1/\overline{X}$.

(b) Based on (a), find an unbiased estimator for λ.

(c) Find the mean squared error (MSE) of $1/\overline{X}$, and the MSE of your estimator from part (b). Which one is smaller?

Problem 5.

Suppose $X_1, \ldots, X_n$ are i.i.d. with p.d.f.

$$f(x; \lambda) = \lambda^2 x e^{-\lambda x}, \quad x > 0,$$

where $\lambda > 0$ is unknown.

(a) Find a maximum likelihood estimator (MLE) for λ.

(b) Find the asymptotic distribution of your MLE from (a).

(c) Find an MLE for $e^{-\lambda}$.

(d) Find the asymptotic distribution of your MLE from (c).

Problem 6.

Let $Y_1, \ldots, Y_n$ be independent, with $Y_i \sim \text{Poisson}(a_i \cdot \nu)$ for some known constants a_i and unknown $\nu > 0$.

(a) Find the joint p.m.f. of $Y_1, \ldots, Y_n$

(b) Find a sufficient and complete statistic for ν.

(c) Determine the unique UMVUE of ν.

(d) Determine the unique UMVUE of ν^2.
