1. There are $k+1$ coins in a box labeled 0 through $k$. Coin $i$, when flipped, will result in “heads” with probability $i/k$, for $i = 0, 1, \ldots, k$. A coin is random selected from the box and repeatedly flipped. ESTIMATE all of the following probabilities for large $k$.

(Hint: You may find it useful at some point in your solution to think about the Beta probability distribution!)

(a) If the first $n$ flips are all heads, find the conditional probability that the $(n+1)$st flip is also heads?

(b) If the first $n$ flips resulted in $r$ heads and $n-r$ tails, show that the probability that the $(n+1)$st flip results in heads is $(r+1)/(n+2)$.

2. Let $U$ be a uniform random variable on the interval $(0,1)$. Let $c$ be a constant such that $0 < c < 1$. Let $V$ be a continuous random variable with some distribution on $(0,1)$ that is independent of $U$.

(a) Show that $\min \left( \frac{U}{c}, \frac{1-U}{1-c} \right)$ has a $\text{uniform}(0,1)$ distribution.

(b) Find

$$P \left( \min \left( \frac{U}{V}, \frac{1-U}{1-V} \right) < c \right).$$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the normal distribution with mean $\mu$ and variance 1. Suppose that we want to estimate $\tau(\mu) = P(X_1 > 0)$. For the following, you may leave your answers in terms of $\Phi(\cdot)$, the cdf for the standard normal distribution.

(a) Show that $X_1 - X_2$ and $X_1 - \bar{X}$ are independent of $\bar{X}$.

(b) Give an unbiased estimator of $\tau(\mu)$. 
(c) Find the UMVUE (uniformly minimum variance unbiased estimator) of $\tau(\mu)$.
(d) Find the MLE (maximum likelihood estimator) of $\tau(\mu)$.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on the interval $(\theta, \theta + 1)$. Suppose that we wish to test the hypothesis $H_0 : \theta = 0$ versus $H_1 : \theta > 0$ using the test

\[
\text{reject } H_0 \text{ if } X_{(1)} > 1 \text{ or } X_{(n)} > c
\]

where $c$ is a constant to be determined.

(a) Find $c$ such that the test will have level of significance $\alpha$.
(b) Find the power function for your test.
(c) Is your test the UMP (uniformly most powerful) test? Explain.

5. Starting at time 0, satellites are launched at times of a Poisson process with rate $\lambda$. Suppose that each satellite, once launched, has a lifetime, independent of all others and of the launch process, that has cdf $F$ and mean $\mu$. Let $X(t)$ be the number of launched and working satellites at time $t$.

(a) Find the distribution of $X(t)$.
(b) Let $t \to \infty$ to show that the limiting distribution is Poisson($\lambda\mu$).

\[
\begin{align*}
\text{Hint 1: Given that } n \text{ satellites have launched, consider writing the surviving satellites at time } t \text{ as a sum of indicators.}
\text{Hint 2: Note that, for a random sample } X_1, X_2, \ldots, X_n \text{ with order statistics } X_{(1)}, X_{(2)}, \ldots, X_{(n)}, \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} X_{(i)}. \\
\text{Hint 3: Recall that, for a nonnegative random variable } X, \ E[X] = \int_{0}^{\infty} P(X > x) \, dx.
\end{align*}
\]