Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2009

Notice. Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work!	1. ___ 2. ___ 3. ___ 4. ___ 5. ___ **TOTAL.** ___

STUDENT NUMBER: ________________

1. Imagine rolling an unbiased die repeatedly and let T_k be the number of rolls until some number appears repeated k times in a row. For example, if the first few rolls of the die were $6, 4, 5, 5, 5, 4, \dots$ then $T_1 = 1$, $T_2 = 4$, and $T_3 = 5$. (Note that T_1 is always 1.) Find a formula that relates $E(T_k)$ to $E(T_{k-1})$ and use it to determine $E(T_k)$ explicitly for all $k \geq 1$.

2. This problem has two separate parts.

   (a) (*Markov's inequality.*) Show that $a^2 \cdot P[|X| \geq a] \leq E(X^2)$, for any random variable X and real-constant $a > 0$.

   (b) Consider random variables $T_1, T_2, T_3, \dots$ i.i.d. Exponential(1) i.e. each has probability density function $f(t) = e^{-t}$, for $t \geq 0$. Consider a certain continuous function $g : [0, +\infty) \to \mathbb{R}$ such that $c := \max\{|g(t)| : t \geq 0\} < +\infty$.
   According to the Law of Large Numbers:

   $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(T_i) = \int_0^\infty g(t)\, e^{-t}\, dt, \text{ with probability 1.}$$

   Given a certain constant $\epsilon > 0$, use part (a) to determine a value of n for which you can guarantee that

   $$P\left[\left|\frac{1}{n}\sum_{i=1}^{n} g(T_i) - \int_0^\infty g(t)\, e^{-t}\, dt\right| < \epsilon\right] \geq 0.99.$$

   Leave your final answer in terms of ϵ and the constant c.

3. Let $0 < \alpha < 1$ be a given quantity and consider $X_1, \dots, X_n$ i.i.d. Uniform$[0, \theta]$, where $0 < \theta \leq 1$ is an unknown parameter.

   (a) Determine a $100(1 - \alpha)\%$ confidence lower-bound for θ in terms of the random variable

   $$Y := \max_{i=1,\dots,n} X_i.$$

   (b) What is the probability that your confidence lower-bound is greater than 1? Leave your final answer in terms of the unknown parameter θ.

(THIS PROBLEM CONTINUES ON THE BACK!)

(c) Motivated by part (b) consider the random variable

$$Z := \begin{cases} Y/(1-\alpha)^{1/n} & , \quad Y \le (1-\alpha)^{1/n}; \\ Y & , \quad Y > (1-\alpha)^{1/n}. \end{cases}$$

Show that Z is a lower-bound for θ with a confidence of at least $100(1-\alpha)\%$.

4. Let $0 < p < 1$ be an unknown parameter and consider a random sample (X_1, X_2) such that $P[X_1 = k_1, X_2 = k_2] = p^2 \cdot (1-p)^{k_1+k_2}$, for $k_1, k_2 \ge 0$ integers. In what follows we will say that a real-valued function $g(p)$ is *good* if there are constants $\alpha_0, \alpha_1, \alpha_2, \dots$ such that $g(p) = p \sum_{k=0}^{\infty} \alpha_k \cdot (1-p)^k$, for all $0 < p < 1$.

   (a) Show that if $g(p)$ is a good function and $\alpha_0, \alpha_1, \alpha_2, \dots$ are like above then α_{X_1} is an unbiased statistic for $g(p)$.

   (b) Find a UMVUE based on (X_1, X_2) for any good function $g(p)$.

   (c) Find the UMVUE based on (X_1, X_2) for $g(p) = p(1+p)$.

5. Consider a fleet of N buses each of which breaks down independently of the others at a rate μ. When a bus brakes down, it is sent for repair to a depot. The mechanic of the depot can only repair one bus at a time and the repair time is always an exponential random variable with mean $1/\lambda$.

   (a) Determine the equilibrium distribution of the number of buses in the repair depot (i.e. undergoing repair or waiting to be repaired).

   (b) If at a given moment all buses are functional, what is the probability that in the next t units of time no bus will break down? Determine this probability explicitly.

   (c) Assume that $N = 1$. If at a given moment all buses are functional, what is the probability that t units of time later no bus will be in the repair depot? Determine this probability explicitly.