Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2008

Notice: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____
Total ____

Student Number ____________________________

1. Let $X_1, X_2, \ldots$ be a sequence of independent exponential random variables with rate λ. Let $N \sim \mathit{Geometric}(p)$ be a geometric random variable that is independent of the X's. Let $X_{(1)}, \ldots, X_{(N)}$ be the order statistics based on a sample of size N.

   (a) Show that
   $$P(X_{(1)} > a) = \frac{pe^{-\lambda a}}{1-(1-p)e^{-\lambda a}}.$$

   (b) Find $\mathsf{E}[X_{(1)}]$.

2. There are n different types of toys you can get when you buy a particular brand of cereal. Suppose that every time that you buy a box of this cereal, it is equally likely that you will get any one of these toys. Let N be the number of boxes that need to be purchased in order to obtain at least one of each type of toy. Show that the expected value of N is
   $$\mathsf{E}[N] = n\sum_{j=1}^{n}\frac{1}{j}.$$

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with rate λ.

   (a) Find the maximum likelihood estimator of λ. Call it $\hat{\lambda}_n$.

   (b) Is $\hat{\lambda}_n$ an efficient estimator of λ? Explain.

   (c) Now suppose that we do not observe the actual values of the data. Instead, it is only known that k, $(0 \le k \le n)$, of the data values are less than or equal to M and that the remaining $n-k$ are greater than M where M is a fixed positive number. Find the MLE of λ in this case.

4. Let X be a single observation taken from the density

$$f(x) = \begin{cases} \frac{2x}{m} & , \quad 0 \le x \le m \\ \frac{2(1-x)}{1-m} & , \quad m < x \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

where m is an unknown parameter.

Fix $0 < \alpha < 1$ and suppose that l and u are such that

$$P(l \le X \le u) = 1 - \alpha$$

and $u - l$ is minimized.

Determine l and u in terms of α and m. Use these results to form a $1 - \alpha$ confidence interval for m in terms of α and the single observation X.

5. A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. In most cases, the fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. However, if the spider and fly are only one unit apart, there is a 50-50 chance that the fly will see the spider and fly away forever. (If he does not fly away, then he moves or does not move as described above.) The initial distance between the spider and the fly is an integer. When the spider and the fly land in the same position, the spider captures the fly. Note that they move at the same instant, so, the fly is not necessarily caught immediately after they are one unit apart since it is possible that either the fly jumps away from the spider and the spider simultaneously jumps towards the fly resulting in them still being one unit apart OR the fly jumps towards the spider and the spider simultaneously towards the fly, resulting in them being one unit apart with their relative positions switched.

   Create an appropriate Markov chain to answer the following questions.

   (a) Suppose that the spider and fly start 2 units apart. What is the probability that the fly escapes?

   (b) Suppose that the spider and fly start 2 units apart. What is the expected number of jumps until the "game" is over. (That is, what is the expected number of jumps until the fly is either eaten or flies away.)

   (c) Suppose that the spider and fly start 2 units apart. What is the expected number of times the fly is in imminent danger before the game is over where "imminent danger" is defined as the spider and fly being only one unit apart?