Name

Show all relevant work.

Notice: Do four of the following five problems. Place an X in the above space opposite the number of the problem that you are NOT submitting for grading.

1. (a) If $Y_1, Y_2, \ldots, Y_n$ are mutually independent random variables with a common Poisson probability mass function, $P(Y_1 = k) = e^{-\lambda} \lambda^k / k!$, find a minimal sufficient statistic for $\lambda$.

(b) Find $E(3Y_1^2)$.

(c) It is observed that the number of breakdowns per day, $Y$, for a certain machine is a Poisson random variable with mean $\lambda$. The daily cost of repairing these breakdowns is given by $C = 3Y^2$. If $Y_1, \ldots, Y_n$ denotes the observed number of breakdowns for $n$ independently selected days, find an MVUE for $E(C)$.

2. A survey of voter sentiment was conducted in four midcity political wards to compare the fraction of voters favoring candidate $A$. Random samples of 200 voters were polled in each of the four wards, with the results as shown in the accompanying table. The numbers of voters favoring $A$ in the four samples can be regarded as four independent
binomial random variables. Construct a likelihood ratio test of the hypothesis that the fractions of voters favoring candidate A are the same in all four wards. Use $\alpha = .05$.

<table>
<thead>
<tr>
<th>Ward</th>
<th>Opinion</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Favor A</td>
<td>76</td>
<td>53</td>
<td>59</td>
<td>48</td>
<td>236</td>
<td></td>
</tr>
<tr>
<td>Do not favor A</td>
<td>124</td>
<td>147</td>
<td>141</td>
<td>152</td>
<td>564</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>200</td>
<td>800</td>
<td></td>
</tr>
</tbody>
</table>

3. Given the linear regression model

$$Y = \beta x + \varepsilon$$

where $Y$ is an observable random variable, $\beta$ is an unknown constant, $x$ is the regressor variable known without error and $\varepsilon$ is an unobservable random variable with mean 0 and unknown variance $\sigma^2$. You are given a set of $n$ independent observations: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

(a) Derive an estimate for $\beta$ by the method of least squares.

(b) Derive the mean and variance for your estimator of $\beta$ in part (a).

(c) Now with the added assumption that the $\varepsilon$’s are normally distributed give a method for testing

$H_0 : \beta = \beta_0$

$H_1 : \beta \neq \beta_0$

Justify your procedure.

4. The random variable $X$ has an exponential density, that is, $f_X(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ so that the moment generating function for $X$ is $\phi_X(t) = 1/(1 - t)$. The random variable $Y$ has the same probability density function as $X$. $X$ and $Y$ are independent.

(a) Find the moment generating function for $-Y$, that is, find $E(e^{t(-Y)}) = E(e^{-\beta Y})$.

(b) Find the moment generating function for $X - Y$.
(c) The random variable \( W \) has a Laplace distribution, that is, \( f_W(w) = e^{-|w|/2} \).

Find the moment generating function for \( W \). Hint: Write

\[
E(e^{tW}) = \left(\frac{1}{2}\right) \int_{-\infty}^{0} e^{tw} e^w \, dw + \left(\frac{1}{2}\right) \int_{0}^{+\infty} e^{tw} e^{-w} \, dw.
\]

(d) What is the probability density function for the random variable \( X - Y \)? Justify your answer.

(e) Find \( E((X - Y)^4) \).

5. \( X_1 \) is a random variable with a geometric distribution having a parameter \( p = 1/3 \), that is, \( X_1 \) is the waiting time until the first success where the success probability is \( p = 1/3 \). \( X_1, X_2, X_3, \ldots, X_{54} \) are 54 mutually independent random variables all with the same probability mass function as \( X_1 \).

(a) Write an exact formula for, but do not evaluate, \( P(153 < \sum_{j=1}^{54} X_j \leq 180) \). Insert the values 1/3, 2/3, and 54 in the correct places in your formula.

(b) Use the Central Limit Theorem to approximate the probability in part (a). You may choose to use, or choose not to use, the continuity correction. Express your answer in terms of the cumulative normal probability distribution \( \Phi( \ ) \).