Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2014

Notice: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____

Total ____

Student Number ________________________

1. Consider $U \sim Uniform(0,1)$ and let R be a continuous random variable with probability density function $f(r) = re^{-r^2/2}$, for $r > 0$. Define:

$$\begin{aligned} X &:= a + b \cdot R\cos(2\pi U) \\ Y &:= c + d \cdot R\sin(2\pi U) \end{aligned}$$

where a, b, c, d are constants such that $b \cdot d \neq 0$.

Assuming that R and U are independent, respond:

(a) Are X and Y independent?

(b) What are the marginal distributions of X and Y?

(c) Let $V \sim Uniform(0,1)$ be independent of (U, R). Determine a function of V that has the same distribution as R, and explain how you could use it to simulate (X, Y) using the random vector (U, V).

2. Let $X_1, X_2, \ldots$ denote an i.i.d. sequence of $Bernoulli(p)$ random variables and consider the binary sequence defined as $Y_i := \Phi(X_i, X_{i+1})$, for $i \geq 1$, where $\Phi : \{0,1\}^2 \to \{0,1\}$ is the indicator function of $(1,1)$ i.e. $\Phi(u,v) = 1$ when $(u,v) = (1,1)$ and $\Phi(u,v) = 0$ otherwise. For $n \geq 1$, determine:

(a) the expectation of $\sum_{i=1}^{n} Y_i$, and

(b) the variance of $\sum_{i=1}^{n} Y_i$.

Next, define q_n as the probability that $\sum_{i=1}^{n} Y_i = 0$. Determine:

(c) a recursion and initial conditions that uniquely determine the sequence $(q_n)_{n\geq 1}$.

3. Consider i.i.d. random variables $X_1, \ldots, X_n$ generated from the *Maxwell density*:

$$f_\theta(x) = \sqrt{\frac{2}{\pi}} \frac{x^2}{\theta^3} e^{-\frac{1}{2}\frac{x^2}{\theta^2}}, \qquad x > 0,\, \theta > 0.$$

Note this family satisfies the "nice" regularity properties that are useful for examining maximum likelihood estimators. This density describes the distribution of speeds of molecules in thermal equilibrium.

   (a) Derive the score function for one observation and use it to find $\mathrm{E}(X^2)$.

   (b) Find the maximum likelihood estimator for θ, $\hat{\theta}_n$.

   (c) Find the asymptotic distribution of $\hat{\theta}_n$.

   (d) Find the UMVUE for estimating $3\theta^2$. Does this estimator achieve the Cramér-Rao lower bound?

4. Let $X_1, \ldots, X_n$ be i.i.d. *Exponential* random variables with rate $\theta > 0$, i.e.

$$f_\theta(x) = \theta e^{-\theta x}, \qquad x > 0.$$

The goal in this problem is to develop a likelihood ratio test for the one-sided alternative $H_0 : \theta = \theta_0$ vs. $H_a : \theta > \theta_0$.

   (a) Find $\sup_{\theta \geq \theta_0} f_\theta(x_1, \ldots, x_n)$.
   HINT: Your answer will depend on a relationship between $\overline{X}$ and θ_0.

   (b) Write down the likelihood ratio Λ_n; is it increasing or decreasing in $\overline{X}$?

   (c) What is the form of the critical region for the likelihood ratio test in terms of $\overline{X}$? You do not need to find the exact region.

5. Consider a cab-stand at an airport where taxis and small groups of customers arrive with rate $\lambda > 0$ and $\mu > 0$, respectively, with $\lambda < \mu$. Suppose that taxis wait no matter how many other taxis are present and depart as soon as a new group solicits them. Moreover, new customers will immediately seek alternative transportation if no taxi is available by the time they arrive.

   Let X_t denote the number of taxis at the cab-stand at time t.

   Assuming that $X = (X_t)_{t \geq 0}$ is a time-homogeneous Markov process, determine:

   (a) the rate matrix of X;

   (b) the stationary distribution π of X;

   (c) the average number of taxis waiting at the cab-stand; and

   (d) the asymptotic fraction of arriving customers that will take a taxi.

   To receive full credit you must simplify your answers in parts (c)-(d).