Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 2008

Notice: Do four of the following five problems. Place an X on the line opposite the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____
Total ____

Student Number ______________________

1. Let $X_1, \ldots, X_n$ be a random sample from Binomial(m, θ) where m is known.

   (a) Show that $T = \sum_{i=1}^n X_i$ is complete and sufficient.

   (b) Using the Lehmann-Scheffe theorem or otherwise, find the uniformly minimum variance unbiased estimator (UMVUE) for $P(X \le 1)$.

2. Suppose that $c_1, c_2, \ldots, c_n$ are known positive constants and that X_i has a gamma distribution with

$$\mathsf{E}[X_i] = 2\theta c_i \qquad \text{and} \qquad Var[X_i] = 2(\theta c_i)^2$$

   for $i = 1, 2, \ldots, n$ and $\theta > 0$. Assume that $X_1, X_2, \ldots, X_n$ are independent.

   (a) Compute the Cramér-Rao lower bound (CRLB) for the variance of all unbiased estimators of θ.

   (b) Find the maximum likelihood estimator for θ. Call it $\hat{\theta}_{ML}$.

   (c) Is $\hat{\theta}_{ML}$ unbiased? Does it achieve the CRLB?

   (d) Find the UMVUE of θ. (Hint: Don't make this too much work.)

   (e) Consider the class of all unbiased estimators for θ of the form $\hat{\theta} = \sum_{i=1}^n d_i X_i$. Find $d_1, d_2, \ldots, d_n$ so that $\hat{\theta}$ and minimizes the mean-squared error $\mathsf{E}[(\theta - \hat{\theta})^2]$ (Hint: Don't make this too much work.)

3. Let X_1, X_2, X_3 be a random sample from the $N(\mu, 1)$ distribution and let Y_1, Y_2, Y_3 be an independent random sample from a $N(0, 1)$ distribution. Suppose that we are unable to observe the individual X's. Consider the random variables defined as $W_i = X_i + Y_i$ for $i = 1, 2, 3$ and $W_4 = X_1 + X_2$.

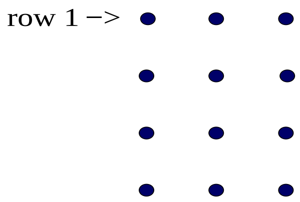
   Let $\mu_1 > 0$ be a known constant.

   (a) Find the best (most powerful) test of size α of $\mu = 0$ versus $\mu = \mu_1$, based only on observing W_1, W_2, and W_3.

(b) Find the best (most powerful) test of size α of $\mu = 0$ versus $\mu = \mu_1$, based on observing W_1, W_2, W_3, and W_4.

(c) Is there a uniformly most powerful test of $\mu = 0$ versus $\mu > 0$ based on W_1, W_2, and W_3? Explain.

4. A subset of 4 dots is selected from a 4 by 3 rectangular array of dots depicted below.



(a) What is the probability that no dot from the first row is selected?

(b) Find the expected number of rows with no selected dots.

(c) Find the variance of the number of rows with no selected dots.

5. Let N(t) be a Poisson counting process with rate λ. Let G(t) be a Gamma process, with parameters α and β, which is defined as follows. G(t) has stationary independent increments, with G(0) = 0 and $G(s+t) - G(s) \sim \Gamma(\alpha t, \beta)$, where $X \sim \Gamma(\alpha, \beta)$ is a non-negative random variable with density $f(x) = \frac{1}{\Gamma(\alpha)} \frac{1}{\beta} (x/\beta)^{\alpha-1} e^{-x/\beta}$.

Define a counting process $M(t)$ by $M(t) = N(G(t))$.

(a) Show that $M(t)$ has stationary and independent increments.

(b) Find $\mathsf{E}[M(s+t) - M(s)]$ and $Var[M(s+t) - M(s)]$.

(c) Show that

$$P(M(s+t) - M(s) = k) = \frac{\Gamma(k + \alpha t)(\lambda\beta)^k}{\Gamma(\alpha t) k! (\lambda\beta + 1)^{k+\alpha t}}.$$