APPM 5450 Applied Analysis II

Proposition 9.6 in Hunter + Nachtergaele

$A \cdot x = \lambda x$, $x \neq 0$, $(A - \lambda I)$ is not invertible.

① Resolvent set $\rho(A)$ } complements.

② Spectrum set $\sigma(A)$

Resolvent operator is $R_\lambda = (\lambda I - A)^{-1}$, so this exists if $\lambda \in \rho(A)$.

Think of a mapping $\lambda \mapsto R_\lambda$

F, $F(\lambda) \in \mathcal{B}(\mathcal{H})$, we know R_λ, if it exists, is bdd (open mapping thm.)

$F : \Omega \to \mathcal{B}(\mathcal{H})$, F is an "operator-valued" function

$\Omega \subseteq \mathbb{C}$ open set.

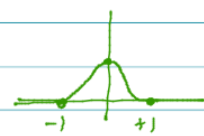
If $F : \Omega \to \mathcal{B}(\mathcal{H})$, Ω open, is analytic at $z_0 \in \Omega$ if $\exists$ $F_n \in \mathcal{B}(\mathcal{H})$,

and $\delta > 0$, s.t. $F(z) = \underbrace{\sum_{n=0}^{\infty} (z - z_0)^n F_n}_{\text{converges w.r.t. the operator norm on } \mathcal{B}(\mathcal{H})}$ $\forall$ $|z - z_0| < \delta$

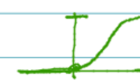
F is holomorphic, analytic, if analytic on all of Ω.

Note: real-analytic $\neq C^\infty$

Ex: $f(x) = \begin{cases} \exp\left(\frac{1}{x^2 - 1}\right) & |x| \leq 1 \\ 0 & \text{else} \end{cases}$



ex. $f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$



$f^{(n)}(\pm 1) = 0$, $f \in C^\infty$, f is not analytic,

f at ± 1, has no convergent power series

PROPOSITION 9.6. $A \in \mathcal{B}(\mathcal{H})$.

Then Ⓐ $\rho(A) \subseteq \mathbb{C}$ is open. Ⓑ R_λ is operator-valued analytic on $\overbrace{\rho(A)}^{\Omega}$

Ⓒ The "exterior disk", $\{\lambda \in \mathbb{C} : |\lambda| > \|A\|\}$ is inside $\rho(A)$.

Proof.

Ⓐ Pick $\lambda_0 \in \rho(A)$.

$(\lambda I - A) = \lambda_0 I - A - (\lambda_0 - \lambda) I$

$= \lambda_0 I - A - (\lambda_0 - \lambda)(\lambda_0 I - A)(\lambda_0 I - A)^{-1}$

Is $\lambda I - A$ invertible? $= (\lambda_0 I - A)\left(I - \underbrace{(\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}}_{T}\right)$

i.e. is $(\lambda_0 I - A)$ invertible ← true since $\lambda_0 \in \rho(A)$

and is $(I - T)$ invertible.

$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ "Neumann Series" (eg. eq 3.13)

by exercise 5.17, this converges if $\|T\| < 1$.

Show $\|T\| < 1$, i.e. $\|(\lambda_0 - \lambda) \cdot \underbrace{(\lambda_0 I - A)^{-1}}_{\text{is bounded}}\| < 1$. Pick $|\lambda - \lambda_0|$ sufficiently small, and $\|T\| < 1$.

Therefore $(\lambda I - A)^{-1} = (I - T)^{-1} / (\lambda_0 I - A)^{-1}$

so $\lambda \in \rho(A)$ if $|\lambda - \lambda_0|$ sufficiently small.

Ⓑ Already proved. Ⓒ Show $\lambda > \|A\| \Rightarrow \lambda \in \rho(A)$ i.e. $R_\lambda = (\lambda I - A)^{-1}$

$R_\lambda = (\lambda I - A)^{-1} = \lambda^{-1}(I - A/\lambda)^{-1} = \lambda^{-1}(I - \cancel{A} T)^{-1}$

$T := A/\lambda$. So $\lambda > \|A\| \Rightarrow \|T\| < 1$. By Neumann Series

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Proposition 9.7 in Hunter + Nachtergaele

Corollary of Prop 9.6: $\sigma(A)$ is closed since $\sigma = \mathbb{C} \setminus \rho$, ρ is open

$\sigma(A) \subseteq B_{\|A\|}(0)$.

Def The spectral radius $r(A)$, $A \in \mathcal{B}(\mathcal{H})$

$r(A) = \sup\{ |\lambda| : \lambda \in \sigma(A) \} = \inf\{ r : |\lambda| \le r \ \forall \lambda \in \sigma(A) \}$

. Corollary of Prop 9.6: $r(A) \le \|A\|$.

PROPOSITION 9.7

$A \in \mathcal{B}(\mathcal{H})$, (1) $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n}$

(2) $A = A^*$, $r(A) = \|A\|$.

Proof.

(1a) Show $\lim \|A^n\|^{1/n}$ exists. Define $\alpha_n = \|A^n\|^{1/n}$

and $\lim \alpha_n$ exists iff $\lim \log(\alpha_n)$ exists, $\log(\alpha_n) = \frac{1}{n} \cdot \underbrace{\log(\|A^n\|)}_{a_n}$

wts $\lim_{n\to\infty} \frac{1}{n} a_n$ exists.

Use sub-multiplication: $\|A^{m+n}\| \le \|A^m\| \|A^n\|$

i.e. $a_{m+n} \le a_m + a_n$. $(*)$

$\Rightarrow a_n \le n \cdot a_1$, and a_n/n is bounded.

Fix m, $\forall n$, write $n = p\cdot m + q$, $q < m$ is residual

$a_n = a_{pm+q} \le a_{pm} + a_q$

$\overset{*}{\le} p \cdot a_m + a_q$, $\frac{a_n}{n} \le \frac{p}{n} a_m + \frac{1}{n} a_q$.

take $n \to \infty$, so $p \to \infty$

$\lim P/n = \lim \frac{P}{p\cdot m + q} = \lim \frac{1}{m + q/p} = \frac{1}{m}$ ($q/p \to 0$)

$\limsup_{n\to\infty} a_n/n \le \limsup_{n\to\infty} \left(P/n \, a_m + \cancel{1/n \, a_q} \right)$ $\to 0$ $\to$ a_q is bounded

$= a_m \cdot \limsup_{n\to\infty} P/n$

$\forall m.$ $= a_m \cdot \lim_{n\to\infty} P/n = a_m/m$.

$\limsup_{n\to\infty} \frac{a_n}{n} \le a_m/m$

Take $\liminf_{m\to\infty}$. $\underbrace{\limsup_{n\to\infty} a_n/n}$ $\le \liminf_{m\to\infty} \underbrace{a_m/m}$.

Same sequence $\Rightarrow \liminf a_n/n = \limsup a_n/n$

$\Rightarrow \lim_{n\to\infty} a_n/n$ exists.

(1b) Use trick: $\lim \|(\lambda A)^n\|^{1/n} = \lambda \cdot \lim \|A^n\|^{1/n}$

(Prop 9.7 continued)

(1b) Aside Neumann Series. $T \in B(X)$, X normed linear space

$$S = \underbrace{I + T + T^2 + \dots}_{\text{Neumann Series}} = \sum_{n=0}^{\infty} T^n$$

Thm (1) If S converges (w.r.t. operator norm) then $(I-T)^{-1}$ exists, and $(I-T)^{-1} = S$

(2) If $\|T\| < 1 \Rightarrow S$ converges.

Generalizing $\frac{1}{1-x} = 1 + x + x^2 + \dots$ (and converges absolutely if $|x|<1$)

Caveats $|x|<1$, or $\|T\|<1$, is sufficient but not necessary for inverse to exist

Ex $x=1$, $\frac{1}{1-x} = \frac{1}{0}$, $(1-x)$ not invertible

Ex $x=-1$, $\frac{1}{1-x} = \frac{1}{2}$, $(1-x)^{-1}$ exists

Define $r = \lim \|A^n\|^{1/n}$

If $r<1$, So $\exists N$ s.t. $\forall n \geq N$, pick $\varepsilon = \frac{1-r}{2}$ so $R = r+\varepsilon < 1$

$\left| \|A^n\|^{1/n} - r \right| < \varepsilon$, $\|A^n\|^{1/n} < r+\varepsilon = R$, So $\underline{\|A^n\| < R^n}$.

$$S = \underbrace{I + A + A^2 + \dots + A^{N-1}}_{\text{bounded}} + \underbrace{A^N + A^{N+1} + \dots}_{S_2}$$

S_2, triangle inequality

$$\|S_2\| \leq \sum_{n=N}^{\infty} \|A^n\| \leq \sum_{n=N}^{\infty} R^n < \infty.$$

So S converges

$(I-A)^{-1}$ exists.

Final Steps

Observe $(\lambda I - A)^{-1} = \lambda^{-1}(I - A/\lambda)^{-1}$, why?

$(\lambda I - A)^{-1} x = y$

$x = (\lambda I - A) y$

$x/\lambda = (I - A/\lambda) y$

$\left[(I - A/\lambda) \cdot \frac{1}{\lambda} \right] x = y$

Define $\tilde{A} = A/\lambda$

If $\underbrace{\lim \|\tilde{A}^n\|^{1/n}}$ $< 1 \Rightarrow (I-\tilde{A})^{-1}$ exists

$\Rightarrow (\lambda I - A)^{-1}$ exists

$\Rightarrow \lambda \in \rho(A)$.

$= \lim \|(A/\lambda)^n\|^{1/n}$

$= |\lambda|^{-1} \cdot \underbrace{\lim \|A^n\|^{1/n}}_{r}$

$|\lambda| > r \Rightarrow \lambda \in \rho(A)$, $r := \lim \|A^n\|^{1/n}$, i.e. $r(A) \leq r$

Conclude $\lim \|A^n\|^{1/n} \geq r(A)$, and in fact, by Laurent Series argument,

$\lim \|A^n\|^{1/n} = r(A)$.