Prop 8.44 and Extensions

APPM 5150 Beck
Feb 19 2018

Book: Prop 8.44 (A) $(x_n) \subseteq \mathcal{H}$ and $x_n \rightharpoonup x$ then $\|x\| \leq \liminf \|x_n\|$

(B) $(x_n) \subseteq \mathcal{H}$ and $x_n \rightharpoonup x$ and $\lim \|x_n\| = \|x\|$ then $x_n \to x$.

More generally, if X is a Banach Space and

$(x_n) \subseteq X$, and $(x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|) \Rightarrow (x_n \to x)$

we say X has the "Kadeck property"

<u>Def</u> A Banach space is uniformly convex if

$$\left(\begin{matrix} \|u_n\| \leq 1 \\ \|v_n\| \leq 1 \end{matrix}, \ \left\| \frac{u_n+v_n}{2} \right\| \to 1 \right) \Rightarrow \|u_n - v_n\| \to 0.$$

<u>Ex</u> All Hilbert spaces are uniformly convex (Parallelogram law)

L^p and ℓ^p, for $1<p<\infty$, are uniformly convex ("Clarkson's Inequalities")

<u>Thm</u> In a Banach space, (A) $(x_n) \subseteq X$, $x_n \rightharpoonup x$ then $\|x\| \leq \liminf \|x_n\|$

<u>proof</u>: Hahn-Banach

<u>Thm</u> Every finite-dimensional space and uniformly convex Banach space has the Kadeck property.

<u>Proof</u> If X is finite dim., $x_n \rightharpoonup x \Rightarrow x_n \to x$.

If X is uniformly convex, wlog let $\|x\|=1$, define $\lambda_n = \max(1, \|x_n\|)$

So by assumption, $\lambda_n \to 1$. Define $y_n = \frac{1}{\lambda_n} x_n$ so $\|y_n\| \leq 1$.

Easy to prove $y_n \rightharpoonup x$ since $x_n \rightharpoonup x$ and $\lambda_n \to 1$. Furthermore, $\frac{1}{2}(y_n + x) \rightharpoonup x$.

Then by (A),

$$1 = \|x\| \leq \underbrace{\liminf \left\|\tfrac{1}{2}(y_n+x)\right\|} \leq \limsup \tfrac{1}{2}\left(\overbrace{\|y_n\|}^{\leq 1} + \overbrace{\|x\|}^{\leq 1}\right) = 1.$$

$$\Rightarrow 1 = \liminf \left\|\tfrac{1}{2}(y_n+x)\right\| \leq \limsup \left\|\tfrac{1}{2}(y_n+x)\right\| = 1$$

So $\lim \left\|\tfrac{1}{2}(y_n+x)\right\| = 1$

and by uniform convexity, $y_n \to x$. $\square$