Prop 8.44 and Extensions

Book: Prop 8.44  (A) \((x_n) \subseteq X\) and \(x_n \rightharpoonup x\) then \(\|x\| \leq \liminf \|x_n\|\)

(B) \((x_n) \subseteq X\) and \(x_n \rightharpoonup x\) and \(\lim \|x_n\| = \|x\|\) then \(x_n \rightharpoonup x\)

More generally, if \(X\) is a Banach Space and

\[(x_n) \subseteq X, \text{ and } (x_n \rightharpoonup x \text{ and } \|x_n\| \to \|x\|) \Rightarrow (x_n \rightharpoonup x)\]

we say \(X\) has the "Kadec property".

Def A Banach space is uniformly convex if

\[
\left( \frac{\|x_n\| \leq 1, \|x_n + x\| \to 1}{\|x_n - x\| \to 0} \right)
\]

Ex All Hilbert-spaces are uniformly convex (Parallelogram law)

\(L^p\) and \(L^\infty\), for \(1 < p < \infty\), are uniformly convex ("Caccioppoli inequality")

Thm In a Banach space, (A) \((x_n) \subseteq X, x_n \rightharpoonup x\) then \(\|x\| \leq \liminf \|x_n\|\)

proof: Hahn-Banach

Thm Every finite-dimensional space and uniformly convex Banach space has the Kadec property.

Proof If \(X\) is finite-dim., \(x_n \rightharpoonup x\) \(\Rightarrow\) \(x_n \rightharpoonup x\).

If \(X\) is uniformly convex, wlog let \(\|x\| = 1\), define \(\lambda_n = \max(1, \|x_n\|)\)

so by assumption, \(\lambda_n \to 1\). Define \(y_n = \frac{x_n}{\lambda_n}, y_n \subseteq X, \|y_n\| \leq 1\).

Easy to prove \(y_n \rightharpoonup x\) since \(x_n \rightharpoonup x\) and \(\lambda_n \to 1\). Furthermore, \(\frac{1}{2}(y_n + x) \rightharpoonup x\).

Then by (A),

\[
1 = \|x\| \leq \liminf \frac{1}{2}\|y_n + x\| \leq \liminf \frac{1}{2}\|y_n\| + \|x\| = 1.
\]

So

\[
\|y_n + x\| = 1
\]

and by uniform convexity, \(y_n \rightharpoonup x\). □