Basic properties of projections
APPM 5450 Spring 2018 Applied Analysis 2

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1 Projections (not necessarily orthogonal)

Let P be a nonzero projection.

1. There is a 1-1 correspondence between projections and direct sums of the vector space X as $X = M \oplus N$ for subspaces M, N.
2. P is a projection if it is an operator $P : X \to X$ that satisfies:
   a) linear (though not necessarily bounded)
   b) $P^2 = P$
3. P need not be bounded[1]
4. P can be bounded (e.g., it always is in finite dimensions). If it is bounded, then
   a) $\|P\| \geq 1$
   b) if $\|P\| = 1$ then in fact P is an orthogonal projection
   c) $\text{ran}(P)$ is closed
5. $x \in \text{ran}(P)$ iff $x = Px$

2 Orthogonal Projections

Let P be a nonzero orthogonal projection on $\mathcal{H}$. In addition to the general properties listed above,

1. There is a 1-1 correspondence between projections and closed subspaces M of $\mathcal{H}$, i.e., with orthogonal direct sums $\mathcal{H} = M \oplus M^\perp$ for closed subspaces M.
2. P is an orthogonal projection if it is an operator $P : X \to X$ that satisfies:
   a) linear (it will turn out to also be bounded)
   b) $P^2 = P$
   c) $P = P^*$
3. P is always bounded, and $\|P\| = 1$
4. $I - P$ is also an orthogonal projection, and $\ker(I - P) = \text{ran}(P)$ and $\text{ran}(I - P) = \ker(P)$ and

[1] See Example 5.7 in our text for how to construct an unbounded linear functional φ on a Hilbert space, and then find some u such that $\varphi(u) = 1$ and define $Px = \varphi(x)u$ (somewhat similar to the ideas just above Example 8.8). Then this P is a unbounded projection.