Comet Mission: 1998P/Willis

Due Thursday, December 7 at 11:59 p.m. on D2L

You must read all instructions on the course webpage in the Projects section, in addition to these instructions (beginning to end!). Failure to do so may result in a loss of points, or possibly no credit for the lab.

1 Introduction

Due to their peculiar appearance in the night sky, comets have been admired and studied for more than 2200 years. It is thought that comets have been largely unperturbed since the formation of the solar system. Thus, scientists use comets to get a glimpse at the conditions of the early solar system. It is also theorized that comets carried a significant portion of water and complex molecules to primordial Earth.

See Figure 1 for a diagram of the large-scale structure of a comet. The most noticeable features of a comet are a loosely packed nucleus, which is comprised of dust, ice, and small rocks; a coma comprised of dust and other molecules; and sometimes dust and ion tails.

In recent years there have been a couple high-profile comet exploration missions: NASA's Deep Impact and ESA's Rosetta. These missions aimed to (more-or-less) directly study the composition of the nucleus of comets. The Deep Impact mission sent a massive (370 kg) impactor into comet 9P/Temple's nucleus and studied the ejected debris; the Rosetta mission landed a (100 kg) lander on the surface of comet 67P/Churyumov-Gerasimenko!

In this lab, you will study various properties of a comet and its gravitational potential field. You will then analyze sending a lander to the surface of the nucleus and an orbiter collecting coma dust while orbiting the nucleus. You must also use the MKS system of units (e.g. m, kg, s, N, J, ...); you don't want to repeat the mistakes made on the Mars Climate Orbiter!

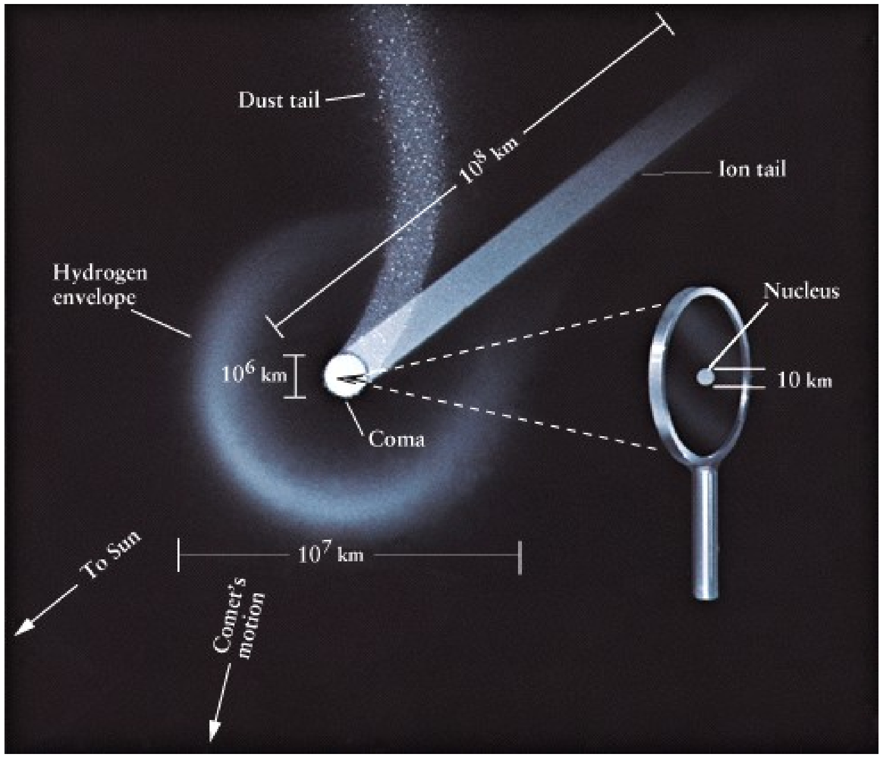

Figure 1: Large-scale structure of a comet. Source: `http://wisp.physics.wisc.edu/astro104/lecture26/F17_20.jpg`.

2 Comet properties

Telescope observations indicate that the nucleus of comet 1998P/Willis is a solid ellipsoid with principal axes $a = 3025\,\mathrm{m}$, $b = 2520\,\mathrm{m}$, and $c = 6050\,\mathrm{m}$:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1.$$

From previous comet missions, we expect the mass density ρ_{comet} to be approximately constant and $\rho_{\text{comet}} = 400\,\mathrm{kg/m^3}$.

1. Use the Mathematica function `ContourPlot3D` to visualize the nucleus. You might consider using the options `BoxRatios->{a/c,b/c,1}, Mesh->None, ContourStytle->{Gray}`. Be sure to label the axes.

2. **Set up and evaluate triple integrals** to find the following quantities. You should consider using `NIntegrate`. In each case, there may exist simple expressions for the desired quantity, but **you must use Calculus 3 techniques**. Remember to report your answers in the MKS system of units.

   (a) The mass of the comet's nucleus, m_{comet}.

(b) The moment of inertia about each coordinate axis (i.e. I_x, I_y, I_z).

(c) The surface area of the comet's nucleus, SA_{comet} (use a double integral for this).

3 Multipole expansion

In order to safely land a spacecraft on the surface of the comet's nucleus, you need an accurate formula for the gravitational potential V around the nucleus. Let $\mathbf{x} = \langle x, y, z \rangle$ be the Cartesian coordinates of space and $\mathbf{r} = \langle \xi, \eta, \zeta \rangle$ be Cartesian integration variables. The gravitational potential of a solid body is given by

$$V(\mathbf{x}) = -\iiint_\Omega \frac{G}{|\mathbf{x} - \mathbf{r}|} \mathrm{d}m(\mathbf{r}) = -\iiint_\Omega \frac{G}{|\mathbf{x} - \mathbf{r}|} \rho(\mathbf{r}) \mathrm{d}V(\mathbf{r}),$$

where $\mathbf{r}$ is the integration variable (not r from cylindrical coordinates!), Ω represents the solid body, and $G = 6.67408 \times 10^{-11}\,\mathrm{Nm}^2/\mathrm{kg}^2$ is the universal gravitational constant (in MKS, V has units J/kg). Note that the gravitational potential is different from the gravitational potential *energy*. We can compute the potential energy of an object of mass m in the field produced by the potential V via $U = mV$.

Writing the above formula in terms of x, y, z and the integration variables ξ, η, ζ, we have

$$V(x, y, z) = -\iiint_\Omega \frac{G}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \rho(\xi, \eta, \zeta) \mathrm{d}V(\xi, \eta, \zeta).$$

The notation $\mathrm{d}V = \mathrm{d}V(\xi, \eta, \zeta)$ is used to make it clear that ξ, η, and ζ are integration variables; the above formula is still just a normal triple integral.

For the comet, this integral cannot be fully evaluated symbolically. However, we can use `NIntegrate` to find a good approximate value for a fixed (x, y, z).

3. Define a function using `NIntegrate` to compute the (true) gravitational potential around the comet. You may want to use the following code snippet:

```
V[x_?NumericQ, y_?NumericQ, z_?NumericQ] := NIntegrate[...];
```

With this function, we can evaluate the potential for any given point (x, y, z), but we cannot take x, y, or z derivatives of this function. For example, to find the force on a particle of mass m due to the nucleus, we would use $\mathbf{F} = -m\nabla V$. So, we need another method of computing V.

An alternative is to use the **multipole expansion** to find an approximation $V_m(\mathbf{x})$ to $V(\mathbf{x})$. The basic idea of the multipole expansion is to expand the true potential as an infinite series. One can then truncate the series, resulting in an approximation of the true potential, much like one could do with a Taylor series. A complete derivation of the multipole expansion is beyond the scope of this project, but we can use it nonetheless.

For the comet, the multipole expansion is given by

$$V(\mathbf{x}) = -\frac{G}{|\mathbf{x}|} m_{\text{comet}} - \frac{G}{|\mathbf{x}|^3} \iiint_\Omega |\mathbf{r}|^2 \frac{3\cos^2\theta - 1}{2} \rho_{\text{comet}}(\mathbf{r}) \mathrm{d}V(\mathbf{r}) + \cdots ,$$

where $\mathbf{r}$ is again the integration variable, and $\theta = \theta(\mathbf{x}; \mathbf{r})$ is the angle between $\mathbf{x}$ and $\mathbf{r}$. It turns out that the first two terms are symbolically integrable, and so we can define the approximation

$$\begin{aligned} V_m(x, y, z) = &-\frac{G}{\sqrt{x^2 + y^2 + z^2}} m_{\text{comet}} \\ &- \frac{G}{(x^2 + y^2 + z^2)^{3/2}} \iiint_\Omega (\xi^2 + \eta^2 + \zeta^2) \frac{3\cos^2(\theta(x, y, z; \xi, \eta, \zeta)) - 1}{2} \rho_{\text{comet}}(\xi, \eta, \zeta) \mathrm{d}V(\xi, \eta, \zeta). \end{aligned}$$

4. Compute $V_m(x, y, z)$ in Mathematica using `Integrate`, and plot $V(0, 0, z)$ and $V_m(0, 0, z)$ over $z \in [c, 2c]$. Be sure to label the axes (with MKS units) and provide a legend for the plot. Note that $V_m(\mathbf{x})$ should be a good approximation of $V(\mathbf{x})$ for $\mathbf{x}$ outside the nucleus (but not inside).

4 Landing on the comet

Once you have the (approximate) gravitational potential, $V_m(\mathbf{x})$, the astrodynamicists on your team can design the orbit of the orbiter and the landing trajectory of the lander. Let's say they design the trajectory of the lander to be

$$\begin{aligned} \mathbf{r}_{\text{lander}}(t) = \langle &-7000 + 0.001t + 3 \times 10^{-6} t^2, \\ &-5000 + 0.001t + 2 \times 10^{-6} t^2, \\ &10000 - 0.05t - 2 \times 10^{-6} t^2 \rangle, \end{aligned}$$

where $t \geq 0$ is the time in seconds after releasing the lander from the orbiter and $\mathbf{r}_{\text{lander}}$ is measured in meters. The lander will maintain this trajectory by firing maneuvering thrusters.

5. Find the time t_{land} when the lander lands on the surface of the comet. Use this to plot the path of the lander along with the nucleus, a green sphere at the release location, and a white sphere at the landing site.

6. What is the initial velocity of the lander? The mass of the lander is $m_{\text{lander}} = 105\,\text{kg}$. What is the initial force on the lander due to the comet's gravity (use V_m)?

7. Next you should determine if the landing speed is safe. The lander has harpoons that will attempt to secure the lander to the soft surface of the nucleus, but they will only work if the lander's landing speed is less than $0.5\,\text{m/s}$. If the lander impacts with too high a speed, it may bounce off the nucleus and become lost in space.

   (a) What is the velocity of the lander at impact if it follows the path $\mathbf{r}_{\text{lander}}(t)$? Does the lander make a safe landing in this case?

   (b) Compute the work done on the lander by the comet's gravity in a few different ways. Note that the work done by gravity will be the same if the lander uses its thrusters or falls freely under the influence of gravity. First, directly compute the work done using a line integral of the force on the lander along the path of the lander; you should use V_m for this. Second, use conservation of energy with V_m, and check that it agrees with the first method. Pages 785-786 of the textbook discuss conservation of energy.

   (c) Use the above calculations to determine the work done by the lander's thrusters in slowing its descent when it follows the path $\mathbf{r}_{\text{lander}}(t)$.

   (d) Now consider what would happen if the lander didn't use its thrusters to control its descent. Instead, it would start at the original release point in orbit with the same initial speed and land in the same spot, but would free-fall to the nucleus. Compute its free-fall landing speed using conservation of energy with V_m. Does the lander make a safe landing in this case?

5 Orbiting in the coma

The astrodynamicists on your team also designed the orbit of the orbiter:

$$\begin{aligned}\mathbf{r}_{\text{orbiter}}(t) = \langle & -1769.42 - 5230.58\cos(0.0001t) - 5362.31\sin(0.0001t), \\ & 1233.56 - 6233.56\cos(0.0001t) + 4499.51\sin(0.0001t), \\ & 4187.62 + 5812.38\cos(0.0001t)\rangle.\end{aligned}$$

Again t is in seconds and $\mathbf{r}_{\text{orbiter}}$ is measured in meters.

8. Add the path of the orbiter (over one period) to your plot of the comet and lander trajectory.

While the lander is studying the surface of the nucleus, the orbiter will collect dust in the coma. The orbiter has a forward looking collector/analyzer with area $A_{\text{det}} = 0.04\,\text{m}^2$ that will capture and analyze dust that falls into it. The mass density of dust in the coma nearby the nucleus is approximately $\rho_{\text{dust}} = 2 \times 10^{-6}\,\text{kg/m}^3$. The velocity field of the dust is given by

$$\mathbf{v}_{\text{dust}}(x, y, z) = 1.7\langle e^{-4\times 10^{-8}x^2}, e^{-4\times 10^{-8}y^2}, e^{-1.6\times 10^{-9}z^2}\rangle\ \text{m/s}.$$

Assume the mass flux of dust is constant across the surface of the collector. Note that the amount of mass collected depends on the relative velocity of the orbiter and dust, $\mathbf{v}_{\text{orbiter}} - \mathbf{v}_{\text{dust}}$, and also the angle between the normal to the collector and the dust velocity. The following integral in t will approximate the amount of dust collected in one orbit:

$$m_{\text{collect}} = A_{\text{det}} \int_0^{T_{\text{orbit}}} \max\left\{0, (\mathbf{v}_{\text{orbiter}}(t) - \mathbf{v}_{\text{dust}}(\mathbf{r}_{\text{orbiter}}(t))) \cdot \mathbf{T}_{\text{orbiter}}(t)\right\} \rho_{\text{dust}}(\mathbf{r}_{\text{orbiter}}(t))\mathrm{d}t,$$

where T_{orbit} is the period of one orbit, $\mathbf{v}_{\text{dust}}(\mathbf{r}_{\text{orbiter}}(t))$ is the dust velocity along the path of the orbiter, $\mathbf{T}_{\text{orbiter}}(t)$ is the unit tangent vector of the orbiter's path, and $\rho_{\text{dust}}(\mathbf{r}_{\text{orbiter}}(t))$ is the density of dust along the orbiter's path.

9. Compute the mass collected in a single orbit. The mass should be about a few grams. You will probably want to use `NIntegrate`.