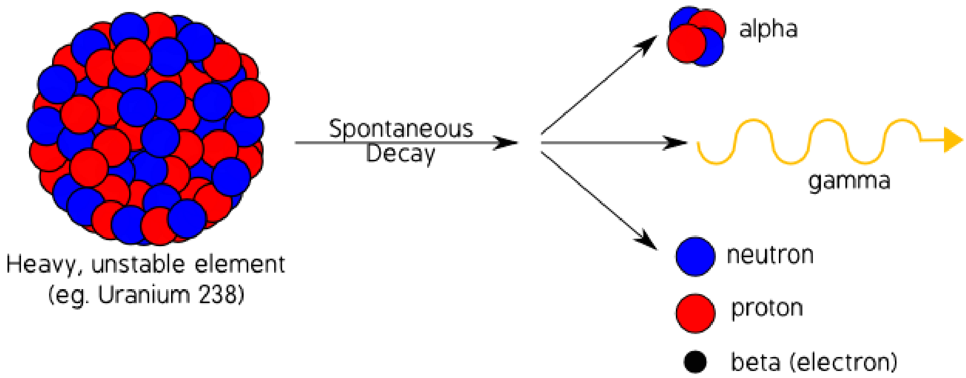
APPM 2360 Project 1
Radioactive Decay

Due: Thursday, October 4, 2018 by 4:59 p.m.
Submit as a PDF to "Assignments" on Canvas



---

1 Introduction

This lab demonstrates the use of first order differential equations to model a naturally occurring phenomenon, radioactive decay. The purpose of this lab is to use numerical solution techniques and analytical methods to study the equations that model radioactive decay chains. The models discussed in this lab will be examined both analytically and graphically.

2 Background

Protons and neutrons are the subatomic particles that form the nuclei of atoms. The number of protons in the nucleus determines the identity of the element (for example, an atom with 92 protons is uranium but an atom with 82 protons is lead). The number of neutrons in the nucleus can vary without changing the identity of the element (for example, uranium-235 contains 92 protons and 143 neutrons but uranium-238 contains 92 protons and 146 neutrons). Certain atomic nuclei such as uranium have combinations of protons and neutrons that are unstable and therefore undergo radioactive decay. During the decay process, particles are ejected from the nucleus (this is radiation) and often the number of protons and neutrons is altered, changing the identity of the element.

An important concept in radioactive decay is half-life. This is the amount of time required for half of the atoms in a sample to decay and it is denoted $t_{1/2}$. The notion of half life has many uses in the modern world. A familiar example is found in the field of radiography, which applies the properties of radioactive decay to develop techniques for inspecting materials or objects without damaging them (these are known as "non-destructive testing (NDT)" methods). NDT methods that you have likely encountered use X-ray radiography – in the form of X-ray machines – to image the internal structure of items such as luggage in the security lines at airports or your teeth when you go to the dentist. To capture clear images, the radiographer must use a specific amount of X-ray radiation. However, radioisotopes decay with time and this decay alters the intensity of the radiation that is produced. Using half-lives, radiographers can determine how intense the radiation source is and adjust the amount of radiation that an item or person is exposed to.

2.1 Mathematical Model

Radioactive decay is a random process, but for very large numbers of atoms we say that the "rate of decay," or the number of atoms that decay per second, is proportional to the number of atoms present. Since the process of radioactive decay is essentially a rate of change, it is natural to model the process with differential equations. To model the simplest type of radioactive decay, we turn to conservation laws. The rate of change of the particles can be described by the following:

$$\text{RATE OF CHANGE} = \text{RATE IN} - \text{RATE OUT} \tag{1}$$

Since no new particles are being created, the RATE IN is zero, and the above equation becomes:

$$\text{RATE OF CHANGE} = -\text{RATE OUT}$$

Now introduce the notation that will be used to describe the radioactive decay process mathematically. Call the element undergoing radioactive decay element A. Next, define the number of atoms present at any given time t as $N_A(t)$. Notice that this function depends on time because the number of atoms must change during the decay process. The number of atoms present is proportional to the rate at which they will decay, but we would like to obtain the equations above. (That is, we want to obtain an equality between the rate of decay and the number of atoms present.) Define the "rate constant" k_A to be the constant of proportionality. The rate of change equations can then be written in our new notation as

$$\frac{d}{dt}N_A(t) = -k_A N_A(t) \tag{2}$$

2.2 Questions

1. Solve (2) analytically for $N_A(t)$ given the initial condition $N_A(0) = A_0$. (Your answer should be expressed symbolically in terms of $N_A(t)$, k_A, A_0, and t.)

2. (a) Use the result of question 1 to solve for k_A. (This is just a rearrangement of terms. Your answer should still be expressed symbolically.)

   (b) Now use the result of 2(a) and the fact that $N_A(t_{A,1/2}) = \frac{1}{2}A_0$ to solve for k_A. (Your answer should still be expressed symbolically.)

   (c) What are the units of the proportionality constant k_A (explain)? Use the result of 2(b) and the half life $t_{A,1/2} = (5\ln 2)$ seconds to find a value for k_A.

3. (a) Suppose you have a sample of element A that initially has $A_0 = 15,000$ atoms. Use Euler's method to solve (2) for the step sizes $\boldsymbol{h_1 = 1}$, $\boldsymbol{h_2 = 0.1}$, and $\boldsymbol{h_3 = 0.01}$. Plot all numerical solutions and the exact solution on a single plot for an appropriate range of t. (Be sure to include a legend.)

   (b) The *absolute error* of a numerical approximation can be defined by

   $$\text{Abs. Error} = |\text{Exact Solution} - \text{Approximate Solution}|$$

   On the same graph, plot the absolute errors of the numerical solutions for the range of t used in part (a). (Plotting with `semilogy` may be useful. Be sure to include a legend.)

(c) How well do the numerical solutions approximate the exact solution? Briefly discuss how changing the step size affects the accuracy of the numerical solution. As the Euler method is made more accurate, the time required to find the numerical solution increases. Discuss the trade-offs between numerical accuracy and numerical efficiency involved in choosing a step size, and decide which of the three step sizes used might give the best balance of accuracy and efficiency.

4. Determine approximately how long it will take until the sample of A has completely decayed away. Explain your reasoning.

3 Decay Chains

We now turn our attention to a decay chain. Most unstable elements do not decay to a stable element in just one step. Instead, the unstable element will decay to a stable isotope in a series of steps called a *decay chain*. During the process of radioactive decay, element A decays to element B which decays to element C. While this process could take many more steps to reach a stable species – for example, the uranium decay series from uranium-238 to lead-206 requires fourteen decay steps – we will say that the element C is stable and does not decay further.

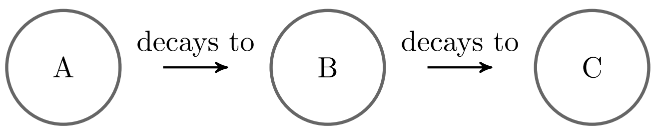


Figure 1: A simple decay chain.

Although the situation can become very complicated, it is still helpful to think of the rate of change for any of these species using the rate of change equation (1) and conservation laws. This gives a template for setting up differential equations for any species in the chain:

$$N'(t) = \text{RATE IN} - \text{RATE OUT} \tag{3}$$

(Keep in mind that the “rate in” and “rate out” expressions in (3) could contain many terms.) Notice that, in this scenario, the decay of species A is the same as in Section 2 and can still be described by (2). This general rate of change model will be used to develop differential equations to describe the other members of the decay chain, $N_B(t)$ and $N_C(t)$.

3.1 Questions

1. Use the rate of change equation (3) and your solution for $N_A(t)$ from Section 2 to set-up a differential equation for $N_B(t)$.

2. Use the integrating factor method to determine the analytical solution for $N_B(t)$ (your answer should be expressed symbolically). To do this, you will need to find a reasonable initial condition for $N_B(0)$. (Hint: how many atoms of B are initially present? Explain.)

3. Use the rate of change equation (3) and the result of question 2 to set-up a differential equation for $N_C(t)$ and then solve analytically for $N_C(t)$ with an appropriate initial condition. Again, explain your choice of $N_C(0)$.

4. For an appropriate range of t, plot $N_B(t)$ and $N_C(t)$ together against t (Note: $t_{B,1/2} = (15 \ln 2)$ seconds). Describe physically the behavior of each species as t approaches infinity. Is this behavior expected?. Why does the graph of $N_B(t)$ have a local maximum?

(The following information should be used to answer questions 5-9 below)

Now let's modify the scheme given above. Usually decay chains are quite large and complicated. Suppose that we only wanted to think about the decay of the last three species of a much larger decay chain. In this scenario, we can solve in a similar way for the decaying species A, B, and C, but we must slightly alter the situation. Now assume that element A is produced in the decay chain at a rate $P > 0$ (for simplicity, assume P is constant), and then A decays to B and B decays to C as before. The differential equations that result from this modified decay chain are very similar to those that resulted from the decay chain in questions 1 and 3 (above).

5. Set up differential equations for $N_A(t)$, $N_B(t)$, and $N_C(t)$ in this modified scenario. The equations should be expressed *symbolically* in terms of $N_A(t)$, k_A, t, $\dots$, etc.

6. Suppose that the production rate of A is $P = 5,000$ atoms/s and the values of the rate constants k_A and k_B are the *same* as in the previous problems. Use Euler's method* and a step size of $h = 0.01$ to solve each of the DEs over the range $t \in [0, 100]$ seconds.

   ($\star$) For these equations, it is possible to implement the Euler method as follows (this is *not* how we usually solve systems of equations numerically): Solve the DE for $N_A(t)$ *first*, solve the DE for $N_B(t)$ *second*, and solve the DE for $N_C(t)$ *last*. This can easily be done in just one `for`-loop, but you may use multiple `for`-loops if you wish.

7. Plot the results for $N_A(t)$ and $N_B(t)$ together against t (be sure to use a legend). Explain how the asymptotic behavior of $N_B(t)$ compares to $N_A(t)$. Is this expected?

8. Plot the result for $N_C(t)$ and describe what is physically happening (especially the behavior as t grows large). Does this make sense? Explain.

9. This discussion of radioactive decay is rather simplistic. Briefly comment on additional assumptions that may affect radioactive decay and chains of decay as examined in this lab. This may require (a little) additional reading on the subject.

4 Report Guidelines

Your group will submit your project write-up on Canvas to the appropriate "Project Assignment" (you can find these under the "assignments" tab in Canvas). Adhere to the following guidelines:

- Do not put off finding a group (you must works in groups of 2-3). You should have a group set-up within one week of the project assignment due date.
- Submit your project in a **pdf** format and submit ALL code used for your project (.nb files for Mathematica, .m files for Matlab, etc). Code may be included in the appendix if you wish. DO NOT submit anything on Canvas as a .zip file. Contents of .zip files will not necessarily be graded.
- Have only ONE group member submit the project. Having multiple people in your group submit the project to Canvas will result in multiple grades, and we will take the LOWEST one.
- Include the names and recitation section numbers of all group members working on the project on the cover page of the report.
- When you submit the report to Canvas, please include each group member's information (name, student number, and section number) in the comments. This allows us to quickly search for a student's report.

Your report needs to accurately and consistently describe the steps you took to answer the posed questions. This report should have the look and feel of a technical paper. Presentation and clarity are very important. Here are some important items to remember:

- Remember: you are to submit a complete report for this project. Documents submitted with numbered responses will be severely penalized.
- Labs must be typed, including all equations in the main body (part of your learning experience is to learn how to use an equation editor). An exception can be made for lengthy calculations in the appendix, which may be hand written (as long as they are neat and clear), and minor labels on plots, arrows in the text, and a few subscripts.
- Write your report in an organized and logical fashion. Section headers such as Introduction, Background, Problem Statement, Calculations, Results, Conclusion, ... are not mandatory but are highly recommended. They not only help you write your report, but help the reader navigate your paper.
- Start with an introduction that describes what you will discuss in the body of the report. A brief summary of important concepts used in your discussion could be helpful here as well. Always introduce relevant equations that will be used or discussed in the report.
- Always include units in your answers
- Always label plots and refer to them in the text.
- You must include any plot that supports your conclusions or gives you insight in your investigations. However, DO NOT use screen-shots of your figures.
- DO NOT include printouts of computer software screens. You simply need to state which software you used in each step and what it did for you.
- The main body of your paper should NOT include lengthy calculations. These should be included in a labeled appendix and should be referred to in the main body.
- DO include the results of any calculations in the body of your report.

- Your report does not have to be long. You need quality, not quantity, of work. Do not omit any important piece of information, but do not feel obligated to add any extras.
- Summarize what you have accomplished in the conclusion. No new information or new results should appear in your conclusion. You should only review the highlights of what your wrote about in the body of the report.