Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2023

Instructions:

Do two of three problems in each section (Prob and Stat).
Place an **X** on the lines next to the problem numbers
that you are **NOT** submitting for grading.

Prob
1. ____
2. ____
3. ____

Do not write your name anywhere on this exam.
You will be identified only by your student number.
Write this number **on each page** submitted for grading.
Show all relevant work!

Stat
4. ____
5. ____
6. ____
Total ____

Student Number ______________________________

Probability Section

Problem 1.

Your electric car's battery fails with rate 0.5/year, and once it fails the repair shop requires an exponential amount of time with rate 4/year to be fixed.

(a) Assume the car battery is a continuous time Markov process with state 0 denoting that it is working, and 1 denoting that it is at the repair shop. Write down the associated rate matrix Q.

(b) Find the probability that your car is working after a *very* long time.

(c) Your car is currently working; you are about to embark on a life-changing road trip for 2 years; find the probability that your car works without failure for the duration of your trip.

(d) Your car is currently working; find, instead, the probability that it is working 2 years from now (note it may fail and be repaired during this time period).

Problem 2.

There are two unrelated parts to this question.

(a) Suppose $X_1, X_2, \ldots$ are a sequence of iid random variables with density $f(x)$ and characteristic function $\mathbb{E}(e^{iuX_j}) = \hat{f}(u)$ for all j. Suppose $N(t), t \geq 0$ is a Poisson process with rate λ that is independent of all X_js. Find the characteristic function of the compound Poisson process

$$Y_t = \sum_{j=1}^{N_t} X_j.$$

(b) A random variable X is said to be infinitely divisible if, for any n, there are n iid random variables $Z_1, \ldots, Z_n$ such that X and $Z_1 + \cdots + Z_n$ have the same distribution. Suppose X and Y are infinitely divisible, show that $aX + bY$ is also infinitely divisible for any real numbers a, b.

Problem 3.

Suppose $X \sim U(-1, 1)$ and $Z \sim U(0, \frac{1}{10})$ and let X and Z be independent. Let $Y = X^2 + Z$. In this problem you will examine the joint behavior of (X, Y).

(a) Find the conditional distribution of $[Y|X = x]$.

(b) Find the full joint pdf of (X, Y). Be sure to specify bounds.

(c) Draw or sketch a picture of the support of the pdf of (X, Y) and guess the correlation between the two variables.

(d) Derive the correlation coefficient between X and Y and check against your guess.

Statistics Section

Problem 4.

Consider the Uniform$(\alpha - \beta/2, \alpha + \beta/2)$ distribution, where $\alpha, \beta \in \mathbb{R}$ and $\beta > 0$. Let $U_1, U_2, \ldots, U_n$ be a sample of i.i.d. random variables from this distribution with $n > 2$.

(a) Calculate the mean squared error (MSE) for two estimators of α called T_n and T_k, where T_n is a function of all n data points in the sample and T_k is a function of only $k < n$ data points. Compare the MSE for T_n and T_k, and show your work.

(b) What is the uniformly minimum variance unbiased estimator (UMVUE) of α? Call this estimator W_n. Justify why W_n is the UMVUE.

(c) You are given a new data point U_{n+1}. What is the UMVUE of α based on the i.i.d. sample $U_1, U_2, \ldots, U_n, U_{n+1}$? Label this estimator W_{n+1}. How does the MSE of W_n compare with the MSE of W_{n+1}? Calculate the probability that $W_n \neq W_{n+1}$.

Problem 5.

Let $Y \sim$ Binomial$(3, p)$, where $p \in [0, 1]$. Consider the null hypothesis $H_0 : p \leq \frac{1}{2}$ versus the alternative hypothesis $H_a : p > \frac{1}{2}$.

(a) What would be the Type I error of this hypothesis test if the rejection region were set to $R = \{2, 3\}$?

(b) If the rejection region were $R = \{2, 3\}$ as above, and the true value for p were $p = \frac{4}{5}$, what would be the Type II error of this test? Show your work. (You may leave your answer as a fraction, i.e., no need to convert your answer to a decimal.)

(c) Construct the uniformly most powerful (UMP) test of size $\alpha = \frac{1}{8}$ for the hypothesis test $H_0 : p \leq \frac{1}{2}$ versus the alternative $H_a : p > \frac{1}{2}$. Justify your answer.

(d) Find the power function for your test from part (c).

(e) For what values of p would the power of your test in part (c) be greater than $\frac{1}{2}$? You may leave your answer arithmetically unsimplified.

Problem 6.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with rate λ and let $Y_1, Y_2, \ldots, Y_n$ be an independent random sample from the exponential distribution with rate $1/\lambda$.

(a) Find the maximum likelihood estimator for $\theta := \lambda^2$, based on all of the available data. Call it $\widehat{\theta}$.

(b) Is $\widehat{\theta}$ an unbiased estimator of θ? (Show your work.)

(c) Find the asymptotic distribution of $\widehat{\theta}$.