Program in Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 11, 2019

Instructions: Do four of the following six problems. Place an X on the lines next to the problem numbers that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number. Write this number **on each page** submitted for grading. Show all relevant work.	1. ____ 2. ____ 3. ____ 4. ____ 5. ____ 6. ____ Total ____

Student Number __________________________

1. Let $N(t), t \geq 0$, be a Poisson process with rate λ that is independent of the sequence of iid random variables $X_1, X_2, \ldots$ with density $f(x)$.

   (a) Find the moment generating function of the compound Poisson process

   $$Y(t) = \sum_{i=1}^{N(t)} X_i.$$

   (b) Find the mean and variance of $Y(t)$.

2. Let $N(t), t \geq 0$, be a Poisson process with rate λ.

   (a) Find the covariance function for N. That is, find $\mathrm{Cov}(N(t_1), N(t_2))$ for arbitrary $0 \leq t_1 \leq t_2 < \infty$.

   (b) Let A_1 be the first arrival time of $N(t)$. Show that the conditional distribution of A_1 given $N(t_0) = 1$ is $U(0, t_0)$.

   (c) Show that $N(t)$ obeys the Markov property in that

   $$\mathbb{E}\big(f(N(t)) \,\big|\, N(u) \text{ for all } 0 \leq u \leq s\big) = \mathbb{E}\big(f(N(t)) \,\big|\, N(s)\big)$$

   for any measurable function f and $s < t$.

3. Consider a simple random walk on $\mathbb{Z}$ starting from zero. Suppose $P(\text{positive step}) = 0.7 = p$ and $P(\text{negative step}) = 0.3 = 1 - p = q$.

   (a) Find the probability that state 2 is reached before state -3.

   (b) Compute the mean number of steps until the random walk reaches state 2 or state -3 for the first time.

   (c) Find the positive step probability p that maximizes the mean number of steps until the random walk reaches state 2 or -3 for the first time starting at 0.

4. Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the uniform$(0, \theta)$ distribution where $1 \leq \theta \leq 2$.

   (a) Find the maximum likelihood estimator (MLE) for θ based on $X_1, X_2, \ldots, X_n$.

   (b) Is this estimator for θ unbiased? Justify your answer.

   (c) Now suppose that we do not get to observe $X_1, X_2, \ldots, X_n$ but instead we observe $Y_1, Y_2, \ldots, Y_n$ where

   $$Y_i = \begin{cases} 1 & , \quad \text{if } 0 \leq X_i \leq 1 \\ 0 & , \quad \text{otherwise} \end{cases}$$

   Find the MLE for θ based on $Y_1, Y_2, \ldots, Y_n$.

   (d) Compute the relative efficiency of these two estimators of θ.

5. An ecologist studying a particular population of grasshoppers measures the height (in cm) of n grasshoppers' initial jump under a replicable set of conditions in the field. A combination of empirical study and theoretical considerations has caused the ecologist to assume that each of these heights $H_1, ..., H_n$ are independent random samples from a Normal distribution with mean μ and standard deviation 1 cm. You should assume this too.

   (a) Show under what conditions on $\{a_1, ..., a_n\}$, where $a_i \in \mathbb{R}$, that $W = \sum_{i=1}^{n} a_i H_i$ will be an unbiased estimator of μ.

   (b) Find the unbiased estimator of this form that has minimum variance. What is the variance of this estimator?

   (c) Is this estimator the UMVUE? Show your work for full credit.

   (d) Now consider that the ecologist wants to estimate the probability that a grasshopper from the same population (under the same field conditions) will jump higher than 10 cm on its initial jump. Find the MLE of this probability. You can express your answer in terms of the c.d.f. of the standard Normal distribution, $\Phi(\cdot)$.

   (e) For her research, the ecologist wants to know if the jumping behavior of the same population of grasshoppers changes during a total solar eclipse. She measured 6 new grasshoppers' initial jumps during the latest eclipse. Recognizing that the field conditions had changed (the eclipse being the only difference in the conditions), she is no longer willing to assume that the grasshoppers' jump heights during an eclipse follow the same $\text{N}(\mu, 1)$ distribution as before. She decides to use the nonparametric sign test under the null hypothesis that an equal number of heights should be above and below μ to test whether the grasshoppers' jumping heights changed during the eclipse. The ecologist's measurements showed that all 6 grasshoppers jumped lower than μ. What is your conclusion about the ecologist's research question?

6. Let $X_1, X_2, \ldots, X_n$ be an iid sample from the distribution with density

   $$f(x|\theta) = \theta x^{\theta - 1}$$

   for $0 \leq x \leq 1$ and $0 < \theta < \infty$.

   (a) Find the expectation and variance of X.

   (b) Find the MLE of θ.

   (c) Find the Cramer-Rao lower bound for the variance of all unbiased estimators of θ.

   (d) Is the MLE of θ the UMVUE (uniformly minimum variance unbiased estimator) of θ? Justify your answer.