Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 2023

Instructions:

Do two of three problems in each section (Prob and Stat).
Place an **X** on the lines next to the problem numbers
that you are **NOT** submitting for grading.

Prob	
1.	____
2.	____
3.	____

Do not write your name anywhere on this exam.
You will be identified only by your student number.
Write this number **on each page** submitted for grading.
Show all relevant work!

Stat	
4.	____
5.	____
6.	____
Total	____

Student Number ____________________________

Probability Section

Problem 1.

Let α_1, α_2, $\beta > 0$ be given constants. Consider a random vector (Z_1, Z_2) taking values in $\{(z_1, z_2) \in \mathbb{R}^2 : z_2 > z_1 > 0\}$ with the joint probability density function

$$f(z_1, z_2) = \frac{z_1^{\alpha_1 - 1}(z_2 - z_1)^{\alpha_2 - 1} e^{-z_2/\beta}}{\beta^{\alpha_1+\alpha_2}\Gamma(\alpha_1)\Gamma(\alpha_2)}, \quad \forall z_2 > z_1 > 0,$$

where $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$, for $x > 0$, is the Gamma function.

(a) What is the probability density function of Z_1?

(b) What is the probability density function of Z_2 given Z_1?

(c) Are Z_1 and Z_2 independent?

(d) Find $\mathbb{E}[Z_1 Z_2]$.

Problem 2.

Let $\{Z_n\}_{n\in\mathbb{N}}$ be i.i.d. random variables with $Z_n \sim$ exponential(1), and $\Lambda : [0, \infty) \to [0, \infty)$ be a deterministic function such that $\int_0^T \Lambda(t)dt < \infty$ for all $T > 0$ and $\int_0^\infty \Lambda(t)dt = \infty$. Consider $\tau_0 := 0$ and random times $\{\tau_n\}_{n\in\mathbb{N}}$ given by

$$\tau_n := \inf\left\{t \geq \tau_{n-1} : \int_{\tau_{n-1}}^t \Lambda(s)ds \geq Z_n\right\} \quad \forall n \in \mathbb{N}.$$

Define the counting process N by $N(t) := n$ for $t \in [\tau_n, \tau_{n+1})$.

(a) Show that for any $t > 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\left(N(t) = n \mid \tau_n\right) = \begin{cases} \exp\left(-\int_{\tau_n}^{t} \Lambda(s)ds\right), & \text{if } \tau_n < t, \\ 0, & \text{if } \tau_n \geq t. \end{cases}$$

(b) Assume $\Lambda(t) \equiv \lambda$ for some $\lambda > 0$. For each $n \in \mathbb{N}$, find the distribution of the random variable $\tau_n - \tau_{n-1}$. Based on this, what is the distribution of τ_n for any $n \in \mathbb{N}$?

(c) Assume $\Lambda(t) \equiv \lambda$ for some $\lambda > 0$. Find $\mathbb{P}[N(t) = n]$ for any $n \in \mathbb{N}$ and $t > 0$.
(**Hint:** Use part (a) and the distribution of τ_n in part (b))

Problem 3.

A video game consists of multiple rounds at different difficulty levels. The game starts with Level 1, and a player will be moved up or down to other levels depending on performance. Specifically, when the player is at Level 1, if he wins (with probability 0.9), he will be moved up to Level 2; if he loses (with probability 0.1), the entire game is over. When the player is at Level 2, if he wins (with probability 0.5), he will be moved up to Level 3; if he loses (with probability 0.5), he will be moved down to Level 1. When the player is at Level 3, if he wins (with probability 0.2), he wins the entire game; if he loses (with probability 0.8), he will be moved down to Level 2.

For each round $n \in \mathbb{N}$, let $X_n \in \{0, 1, 2, 3, 4\}$ denote the current level at round n. Here, $X_n = 0$ means "game over" and $X_n = 4$ means "winning the entire game".

(a) The evolution of X can be described using a Markov chain. Write down the transition matrix P of this Markov chain. Which states are recurrent? Which states are transient?

(b) Let N denote the total number of rounds a player plays before "game over" or "winning the entire game". Find $\mathbb{E}[N]$.

A close examination of a game developer reveals that the time needed for a player to complete one round of the game is exponential distributed with rate 0.3/minute, 0.4/minute, and 0.2/minute, when the round is at Level 1, Level 2, and Level 3, respectively.

Now, for each time $t \geq 0$, let $Y_t \in \{0, 1, 2, 3, 4\}$ denote the current level at time t. Again, $Y_t = 0$ means "game over" and $Y_t = 4$ means "winning the entire game".

(c) The evolution of Y can be described using a (continuous-time) Markov chain. Write down the generator (i.e., rate matrix) Q of this Markov chain.
(**Hint:** Q should reflect the exponential holding times and the transition probabilities in (a)).

(d) Let T denote the total amount of time a player spends before "game over" or "winning the entire game". Find $\mathbb{E}[T]$.

Statistics Section

Problem 4.

Let $X_1, \ldots, X_n$ be a set of independent and identically distributed random variables with common pdf

$$f_\theta(x) = \frac{\theta}{x^2}, \quad 0 < \theta \leq x < \infty$$

(a) What is a sufficient statistic for θ?

(b) Find the MLE of θ.

(c) Show the method of moments estimator of θ does not exist.

Problem 5.

Suppose we have two independent sets of samples $X_1, \ldots, X_n$ which are i.i.d. exponential(mean=θ) and $Y_1, \ldots, Y_m$ which are i.i.d. exponential(mean=μ) (note in this notation we assume θ and μ are the means, *not* the rates).

(a) Find the likelihood ratio test for $H_0 : \theta = \mu$ versus $H_a : \theta \neq \mu$.

(b) Show that the test in part (a) can be written based on the statistic

$$T = \frac{\sum_i X_i}{\sum_i X_i + \sum_j Y_j}.$$

(c) Note that $T \in [0, 1]$ and, based on (b), show that the LRT statistic is a unimodal function of T. If the LRT λ is rejected for $\lambda \leq c$ for a constant c, find the equivalent rejection region in terms of T.

Problem 6.

Suppose $X_1, \ldots, X_n$ are i.i.d. samples from a $N(\theta, 1)$ distribution.

(a) Find a complete, sufficient statistic for θ.

(b) Based on (a), find the UMVUE for θ^2.

(c) Show that your estimator from (b) does not achieve the Cramér-Rao lower bound (you can use the useful fact that $\text{Var}(\overline{X}^2) = 2/n^2 + 4\theta^2/n$).
