Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2018

Instructions: Do four of the following five problems. Place an X next to the number of the problem that you are **NOT** submitting for grading. Please do not write your name anywhere on this exam. You will be identified only by your student number, given below and on each page submitted for grading. Show all relevant work.

1. ____
2. ____
3. ____
4. ____
5. ____

Total ____

Student Number ____________________________

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function
$$f(x;\theta) = \sqrt{\frac{\theta}{2\pi x^3}} \exp\left(-\frac{\theta}{2x}\right) I_{(0,\infty)}(x)$$
for some parameter $\theta > 0$.
   (a) Find the MLE (maximum likelihood estimator) for θ.
   (b) Find the Cramér-Rao lower bound for the variance of all unbiased estimators of θ.
   (c) Find the asymptotic distribution of your MLE from part (a).
   (d) Find the UMVUE (uniformly minimum variance unbiased estimator) for θ. (Assume that $n \geq 3$.)

2. Suppose that we have four identical (possibly biased) coins, each with the same probability p of coming up "Heads" on a toss.

   Toss coin 1 repeatedly until the first "Heads" occurs. Repeat this experiment with coins 2, 3, and 4.

   For $i = 1, 2, 3, 4$, let X_i denote the number of "Tails" counted for coin i before the first "Heads" occurs.

   (a) Based on the data X_1, X_2, X_3, X_4, construct the UMP (uniformly most powerful) test of size $\alpha = 3/16$ of
$$H_0 : p \leq 1/2 \qquad \text{versus} \qquad H_1 : p > 1/2.$$
   (b) Find the power function for your test from part (a).
   (c) Suppose that the first and third coins come up tails on their first toss. Can you perform your test from part (a)? If so, give its conclusion.

3. Let X and Y be independent continuous random variables with $\mathbb{E}Y = 0$ and $r \geq 1$.

   (a) Show that $X + Y$ is more dispersed than X in that
   $$\mathbb{E}|X|^r \leq \mathbb{E}|X + Y|^r.$$
   (Hint: use Jensen's inequality).

   (b) If $\mathbb{E}Y = \mu_Y$ then show
   $$\mathbb{E}|X + \mu_Y|^r \leq \mathbb{E}|X + Y|^r.$$

   (c) Find a counterexample for the case that X and Y are not independent.

4. There are three related parts to this problem. Consider a space with probability measure P and consider a sequence of events $A_1, A_2, \ldots$. You may freely use the following results:

   - If $A_n \supset A_{n+1}$ for all n then
   $$P\left(\cap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$
   - If $A_n \subset A_{n+1}$ for all n then
   $$P\left(\cup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

   (a) Define $\limsup_n A_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m$. Give an interpretation to this quantity in words.

   (b) The first Borel-Cantelli lemma states that for an arbitrary sequence of events $A_1, A_2, \ldots$,
   $$\sum_{i=1}^{\infty} P(A_i) < \infty \quad \text{implies} \quad P\left(\limsup_n A_n\right) = 0.$$
   Prove it.

   (c) Now suppose the events $A_1, A_2, \ldots$ are independent. The second Borel-Cantelli lemma states the converse,
   $$\sum_{i=1}^{\infty} P(A_i) = \infty \quad \text{implies} \quad P\left(\limsup_n A_n\right) = 1.$$
   Prove it. (Hints: first consider the complement of the event, then simplify and use $1 - x \leq \exp(-x)$).

5. The following is a very simplified model for growth of a skin cancer.

   A skin cell is located at each point of the integer lattice $\mathbb{Z}^2$. At any point in time, each cell's cancer status is either benign (B) or malignant (M).

   - A benign cell will live for an exponential amount of time with rate λ_B, at which time it will divide into two cells. One cell will live on the lattice in the place of the original cell and the other will replace one of the four nearest neighbor cells with probability $1/4$ each.
   - A malignant cell will live for an exponential amount of time with rate λ_M, at which time it will divide into two cells. One cell will live on the lattice in the place of the original cell and the other will replace one of the four nearset neighbor cells with probability $1/4$ each.

   In both cases, the replaced cell leaves the system.

   Let $X(t)$ be the number of M-cells at time t. Assume that $X(0) = 1$.

   (a) Let $N(t)$ be the number of pairs of neighboring cells with different cancer status at time t. Give the rates of a one unit increase and decrease for $X(t)$ in terms of $N(t)$. Is $\{X(t)\}$ a Markov chain? Explain.

   (b) Let X_n be the discrete time embedded process immediately after cell divisions. Show that $\{X_n\}$ is a Markov chain and give its transition probalities.

   (c) Find the probability that the cancer dies out as a function of the "carcinogenic advantage" $r := \lambda_M/\lambda_B$.