Robust Maximization of Asymptotic Growth under Covariance Uncertainty

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The Question

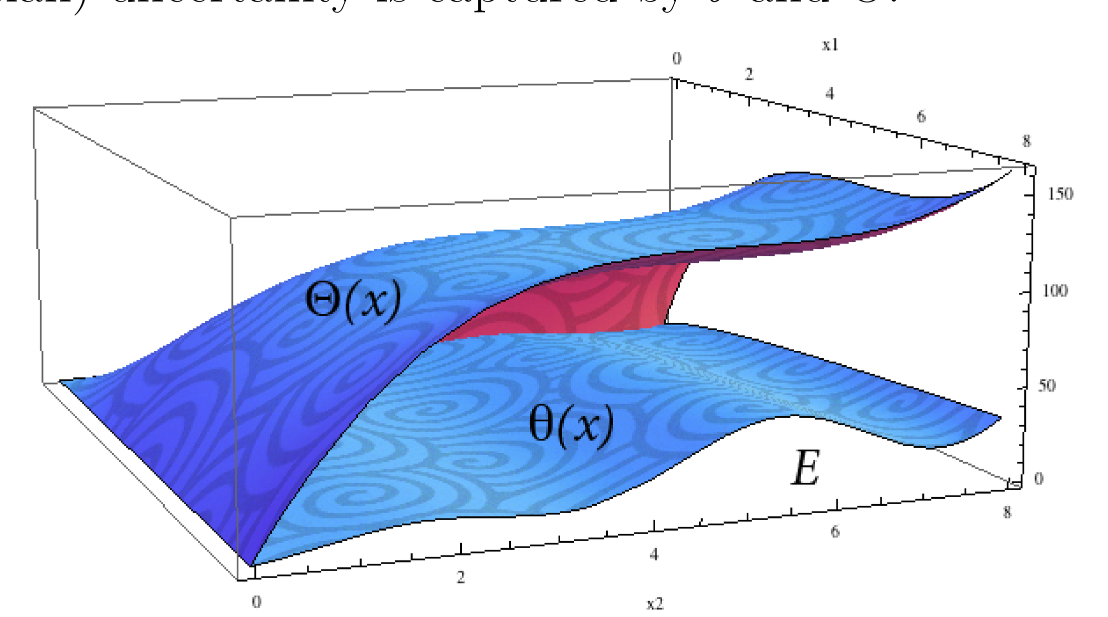
How to maximize the growth rate of one's wealth when **precise covariance structure** of the underlying assets **is not known**?

The Set-up

Let $E \subset \mathbb{R}^d$ be an open connected set, and $\mathbb{S}^d$ be the set of $d \times d$ symmetric matrices.

- $\boldsymbol{X}$: price process of d assets taking values in E.
- $\boldsymbol{\theta, \Theta : E \mapsto (0, \infty)}$ are functions in $C^{0,\alpha}_{\text{loc}}(E)$, and satisfy $\theta < \Theta$ in E.
- $\boldsymbol{\mathcal{C}}$: set of functions $c : E \mapsto \mathbb{S}^d$ s.t. for any $x \in E$,
$$\theta(x) I_d \le c(x) \le \Theta(x) I_d.$$

Remark. Each $c \in \mathcal{C}$ represents a possible covariance structure that might materialize. The (Knightian) uncertainty is captured by θ and Θ.



- $\boldsymbol{(L^{c(\cdot)} f)(x)} := \frac{1}{2}\sum_{i,j=1}^d c_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x)$
$= \frac{1}{2}\text{Tr}[c(x) D^2 f(x)]$.
- $\mathbb{Q}^c$: the solution to the (generalized) martingale problem on E for the operator $L^{c(\cdot)}$.
- $\boldsymbol{\Pi^c} := \{\mathbb{P} \mid \mathbb{P} \ll_{\text{loc}} \mathbb{Q}^c,\ X \text{ doesn't explode } \mathbb{P}\text{-a.s.}\}$
- $\boldsymbol{\Pi} := \cup_{c \in \mathcal{C}} \Pi^c$.
- $\boldsymbol{\pi \in \mathcal{V}}$ (admissible trading strategy): predictable process s.t. the following holds for all $c \in \mathcal{C}$:
  (i) π is X-integrable under $\mathbb{Q}^c$;
  (ii) $V_t^\pi := 1 + \int_0^t \pi_s' dX_s > 0$ $\mathbb{Q}^c$-a.s., for all $t \ge 0$.
- **Asymptotic growth rate** of V^π under $\mathbb{P}$:
$\boldsymbol{g(\pi; \mathbb{P})}$
$$:= \sup\left\{\gamma \in \mathbb{R} \;\middle|\; \mathbb{P}\text{-}\liminf_{t\to\infty} (\log V_t^\pi / t) \ge \gamma \ \mathbb{P}\text{-a.s.}\right\}.$$
$(\approx \sup\left\{\gamma \in \mathbb{R} \mid V_t^\pi \ge e^{\gamma t} \text{ as } t \text{ large } \mathbb{P}\text{-a.s.}\right\})$

Our Goal

Choose an $\pi^* \in \mathcal{V}$ s.t. V^{π^*} attains the rate
$$\sup_{\pi\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi} g(\pi; \mathbb{P})$$
uniformly over all $\mathbb{P}$ in Π (or at least in a large enough subset Π^* of Π).

When $c \in \mathcal{C}$ is fixed...

For any $D \subset E$ and $\lambda \in \mathbb{R}$, we consider
$$H^c_\lambda(D) := \{\eta \in C^2(D) \mid L^{c(\cdot)}\eta + \lambda\eta = 0,\ \eta > 0\},$$
and define the principal eigenvalue for $L^{c(\cdot)}$ on D as
$$\lambda^{*,c}(D) := \sup\{\lambda \in \mathbb{R} \mid H^c_\lambda(D) \ne \emptyset\}.$$

In [3], the authors take $\eta^{*,c} \in H^c_{\lambda^{*,c}(E)}(E)$ and define
$$\Pi^{*,c} := \left\{\mathbb{P} \in \Pi^c \;\middle|\; \mathbb{P}\text{-}\liminf_{t\to\infty} \frac{\log \eta^{*,c}(X_t)}{t} \ge 0 \ \mathbb{P}\text{-a.s.}\right\}.$$
They show that

- $\Pi^{*,c}$ is large enough to include all the probabilities in Π^c under which X is stable.
- $\lambda^{*,c}(E) = \sup_{\pi\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^{*,c}} g(\pi; \mathbb{P}) = \inf_{\mathbb{P}\in\Pi^{*,c}} \sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P})$.
- $\pi_t^{*,c} := e^{\lambda^{*,c}(E)t} \nabla \eta^{*,c}(X_t) \in \mathcal{V}$ satisfies
$$g(\pi^{*,c}; \mathbb{P}) \ge \lambda^{*,c}(E),\ \forall\, \mathbb{P} \in \Pi^{*,c}.$$

When $c \in \mathcal{C}$ is NOT fixed...

Recall **Pucci's operator**: given $0 < \lambda \le \Lambda$,
$$\mathcal{M}^+_{\lambda,\Lambda}(M) := \sup_{A\in\mathcal{A}(\lambda,\Lambda)} \text{Tr}(AM),\ \forall\, M \in \mathbb{S}^d,$$
where $\mathcal{A}(\lambda, \Lambda)$ denotes the set of matrices in $\mathbb{S}^d$ with eigenvalues lying in $[\lambda, \Lambda]$.

Define the operator $F : E \times \mathbb{S}^d \mapsto \mathbb{R}$ by
$$F(x, M) := \frac{1}{2} \sup_{A\in\mathcal{A}(\theta(x),\Theta(x))} \text{Tr}(AM).$$
For any $D \subset E$ and $\lambda \in \mathbb{R}$, we consider
$$H_\lambda(D) := \{\eta \in C^2(D) \mid F(x, D^2\eta) + \lambda\eta \le 0,\ \eta > 0\},$$
and define the principal eigenvalue for F on D as
$$\lambda^*(D) := \sup\{\lambda \in \mathbb{R} \mid H_\lambda(D) \ne \emptyset\}.$$

Now, by using the arguments in [3] and the relation $\lambda^*(E) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(E)$, we obtain

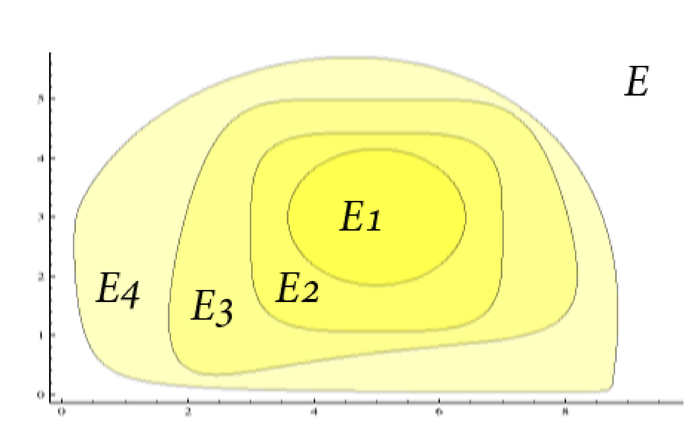
Main Result

Take $\eta^* \in H_{\lambda^*(E)}(E)$. Define
$$\pi_t^* := e^{\lambda^*(E)t} \nabla \eta^*(X_t) \quad \forall\, t \ge 0,$$
and set
$$\Pi^* := \left\{\mathbb{P} \in \Pi \;\middle|\; \mathbb{P}\text{-}\liminf_{t\to\infty} \frac{\log \eta^*(X_t)}{t} \ge 0 \ \mathbb{P}\text{-a.s.}\right\}.$$
Then, we have

- Π^* is large enough to include all the probabilities in Π under which X is stable.
- $\lambda^*(E) = \sup_{\pi\in\mathcal{V}} \inf_{\mathbb{P}\in\Pi^*} g(\pi; \mathbb{P}) = \inf_{\mathbb{P}\in\Pi^*} \sup_{\pi\in\mathcal{V}} g(\pi; \mathbb{P})$.
- $\pi^* \in \mathcal{V}$ and $g(\pi^*; \mathbb{P}) \ge \lambda^*(E)$ for all $\mathbb{P} \in \Pi^*$.

Proving "$\lambda^*(E) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(E)$"

Assume: there exist $\{E_n\}_{n\in\mathbb{N}}$ of bounded open convex subsets of E s.t. ∂E_n is of $C^{2,\alpha}$, $\bar{E}_n \subset E_{n+1}$ $\forall\, n \in \mathbb{N}$, and $E = \cup_{n=1}^\infty E_n$.



Sketch of proof:

1. On each E_n, show the existence of a positive viscosity solution η_n (by using [5]) to
$$F(x, D^2\eta) + \lambda^*(E_n)\eta \le 0.$$
2. Show that η_n is actually smooth (using [6]).
3. Show $\lambda^*(E_n) = \inf_{c\in\mathcal{C}} \lambda^{*,c}(E_n)$.
   (i) $\le$: Use a maximum principle related to F.
   (ii) $\ge$: Use the theory of *continuous selection* in [1] to construct $\{c_m\}_{m\in\mathbb{N}} \subset \mathcal{C}$ s.t.
$$\lambda^*(E_n) \ge \liminf_{m\to\infty} \lambda^{*,c_m}(E_n).$$
4. Show $\lambda^*(E) = \lambda_0 := \lim_{n\to\infty} \lambda^*(E_n)$.
   (i) $\le$: obvious from definitions.
   (ii) $\ge$: Prove a Harnack inequality for F, and use it to show η_n converges uniformly on E to some $\eta^* \in H_{\lambda_0}(E)$.
5. Since $\lambda^{*,c}(E) = \inf_{n\in\mathbb{N}} \lambda^{*,c}(E_n)$ for each $c \in \mathcal{C}$ (by [4]), we have
$$\inf_{c\in\mathcal{C}} \lambda^{*,c}(E) = \inf_{c\in\mathcal{C}} \inf_{n\in\mathbb{N}} \lambda^{*,c}(E_n) = \inf_{n\in\mathbb{N}} \inf_{c\in\mathcal{C}} \lambda^{*,c}(E_n)$$
$$= \inf_{n\in\mathbb{N}} \lambda^*(E_n) = \lambda^*(E).$$

Conclusions and Outlook

Among an appropriate class $\mathcal{C}$ of covariance structures, we characterize the largest possible robust asymptotic growth rate as the principle eigenvalue $\lambda^*(E)$ of the fully nonlinear elliptic operator F, and identify the optimal trading strategy in terms of $\lambda^*(E)$ and the associated eigenfunction.

The covariance uncertainty we consider is similar to the "Knightian uncertainty" formulated in [2], in the sense that the constraint on covariance is Markovian. The latter, however, is more general as it allows the covariance itself to be *non-Markovian*. It is of interest to generalize our results to the case with non-Markovian covariances, which would lead to eigenvalue problems for *path-dependent* PDEs.

For Further Information

- Yu-Jui Huang is available at jayhuang@umich.edu
- Preprint of our paper can be downloaded from www.arxiv.org/abs/1107.2988
- This poster can be downloaded from http://www-personal.umich.edu/∼jayhuang

Selected References

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