Theorem $\pi 1$:
For an irreducible, positive recurrent, aperiodic Markov chain,
\[ \lim_{n \to \infty} p_{ij}^{(n)} \]
exists and is independent of $i$.
(Recall that we have shown that any limiting distribution is stationary.)

Theorem $\pi 2$:
Suppose that a Markov chain defined by the transition probabilities $p_{ij}$ is irreducible, aperiodic, and has stationary distribution $\pi$. Then for all states $i$ and $j$,
\[ p_{ij}^{(n)} \to \pi_j, \quad \text{as } n \to \infty. \]

Theorem $\pi 3$:
For an irreducible, positive recurrent Markov chain, a stationary distribution $\pi$ exists, is unique, and satisfies
\[ \pi_i = \frac{1}{E_i[T_i]}. \]

Theorem $\pi 4$:
For an irreducible Markov chain, a stationary distribution exists if and only if all states are positive recurrent. In this case the stationary distribution is unique.

Note that these Theorems overlap quite a bit. In fact, Theorem $\pi 3$ is just one half of the “if and only if” in Theorem $\pi 4$. Even so, I felt that it was useful to state them in this way.
For Proof of Theorem $\pi 2$:

Lemma ($\pi 2$): If a stationary distribution $\pi$ exists, then all states $j$ that have $\pi_j > 0$ are recurrent.

Proof: We want to show that $g_j = P(T_j < \infty | X_0 = j) = 1$, where $T_j = \min\{n \geq 1 : X_n = j\}$.

Let $N_j$ be the number of visits to state $j$ at times $\geq 1$. Then

$$N_j = \sum_{n=1}^{\infty} I_{\{X_n = j\}}.$$

So,

$$E_i[N_j] = E_i[\sum_{n=1}^{\infty} I_{\{X_n = j\}}] = \sum_{n=1}^{\infty} E_i[I_{\{X_n = j\}}] = \sum_{n=1}^{\infty} P_i(X_n = j) \sum_{n=1}^{\infty} P(X_n = j | X_0 = i) = \sum_{n=1}^{\infty} p_{ij}^{(n)}.$$

Consider the quantity $\sum_i \pi_i E_i[N_j]$:

$$\sum_i \pi_i E_i[N_j] = \sum_i \pi_i \sum_{n=1}^{\infty} p_{ij}^{(n)} = \sum_{n=1}^{\infty} \sum_i \pi_i p_{ij}^{(n)} = \sum_{n=1}^{\infty} \pi_j = \infty$$

since $\pi_j > 0$.

Recall in class where we showed that, starting at state $i$, the expected number of returns to $i$ at times $\geq 1$ was $g_i/(1-g_i)$. We can similarly show that the expected number of visits to state $j$, starting from $i$ is $g_{ij}/(1-g_j)$ where $g_{ij} = P(T_j < \infty | X_0 = i)$.

Then, we have

$$\infty = \sum_i \pi_i E_i[N_j] = \sum_i \pi_i g_{ij} / (1-g_j) \leq \sum_i \pi_i 1 / (1-g_j)$$

since $g_{ij} \leq 1$. So,

$$\infty = \frac{1}{1-g_j} \sum_i \pi_i = \frac{1}{1-g_j} \cdot 1 = \frac{1}{1-g_j}$$

which implies that $g_j = 1$, as desired. □
For Proof of Theorem $\pi 2$ Continued:

Define, for any state $i$, the set

$$A_i = \{ n : p_{ii}^{(n)} > 0 \}.$$ 

Note then that the period of $i$, which we have denoted by $d_i$ is the greatest common divisor of all of the elements in $A_i$.

**Claim:** If $i$ has period 1, then there is a number $K$ such that, for all $n \geq K$, $n$ is in $A_i$.

**Partial Proof of Claim:** We will quote, without proof, a result from number theory that says

“If the greatest common divisor of a set $A_i$ is 1, then there are integers $i_1, i_2, \ldots, i_m$ in $A_i$ and positive or negative integer coefficients $c_1, c_2, \ldots, c_m$ such that $c_1 i_1 + c_2 i_2 + \cdots + i_m c_m = 1$.”

in order to show that $A_i$ contains two consecutive integers. Then we will show that $A_i$ containing two consecutive integers gives us the result we want.

**Proof Step 1:**

If $i$ has period 1, then, by definition, the greatest common divisor of the elements in $A_i$ is 1. So, by the claim quoted above, there are integers $i_1, i_2, \ldots, i_m$ in $A_i$ and positive or negative integer coefficients $c_1, c_2, \ldots, c_m$ such that $c_1 i_1 + c_2 i_2 + \cdots + i_m c_m = 1$.

Let $j$ be the number of positive coefficients and let $k$ be the number of negative coefficients. (So $j + k = m$.) Let $s_1, s_2, \ldots, s_j$ be the subscripts of the $i$’s in $i_1, i_2, \ldots, i_m$ with positive coefficients and let $t_1, t_2, \ldots, t_k$ be the subscripts of the $i$’s with negative coefficients. Define $a_l = c_{s_l}$ for $l = 1, 2, \ldots, j$ and $b_l = -c_{t_l}$ for $l = 1, 2, \ldots, k$.

Then, we have

$$a_1 i_{s_1} + \cdots + a_j i_{s_j} = b_1 i_{t_1} + \cdots + b_k i_{t_k} + 1.$$ (1)
For Proof of Theorem $\pi 2$ Continued:

Note that any positive linear combination of elements in $A_i$ is also in $A_i$. For example, if you can, with positive probability, go from state $i$ to state $i$ in 3 steps and you can also go in 7 steps, then 3 and 7 are numbers in $A_i$. Furthermore, $4 \cdot 3 + 2 \cdot 7$, for example, is also in $A_i$ since you can go from $i$ to $i$ in 3 steps, then another 3 steps, then another 3 steps, then another 3 steps, then in 7 steps, and then another 7 steps.

So, (1) shows us two consecutive integers in $A_i$.

Proof Step 2:

We have now shown that there are two consecutive integers, say $k$ and $k + 1$ in $A_i$. We now wish to conclude that this implies that all integers, after some point are in $A_i$.

If $k$ and $k + 1$ are in $A_i$, then so are

$$2k, \ 2k + 1, \ \text{and} \ 2k + 2$$

since we can go from $i$ to $i$ in $k$ steps plus another $k$ steps or in $k$ steps plus another $k + 1$ steps, or in $k + 1$ steps plus another $k + 1$ steps.

Since $2k, 2k + 1, 2k + 2$ are in $A_i$, then so are

$$4k, \ 4k + 1, \ 4k + 2, \ 4k + 3, \ \text{and} \ 4k + 4.$$

as these are all the distinct sums of pairs from $2k, 2k + 1, 2k + 2$.

Continuing, we get that $k$ and $k + 1$ in $A_i$ implies that

$$jk, jk + 1, \ldots, jk + j$$

are in $A_i$ for any positive integer $j$.

For $j \geq k - 1$, these blocks of numbers included in $A_i$ will start to overlap, thereby leaving no gaps in the remaining sequence of integers included in $A_i$. □
Proof of Theorem $\pi 2$:

We are now ready to prove the theorem.

1. Let $S$ denote the state space and let $\{X_n\}$ and $\{Y_n\}$ denote two independent copies of the Markov chain. Consider the bivariate Markov chain $\{(X_n, Y_n)\}$ on $S^2 = S \times S$ and let the transition probabilities be denoted by

$$p(i_x, i_y; j_x, j_y).$$

Note that, by definition of $\{X_n\}$ and $\{Y_n\}$ and their independence,

$$p(i_x, i_y; j_x, j_y) = p_{i_x,j_x} \cdot p_{i_y,j_y}.$$

We are going to show that

$$|P(X_n = j) - P(Y_n = j)| \to 0, \quad \text{as } n \to \infty$$

(2) regardless of the starting values of $\{X_n\}$ and $\{Y_n\}$. So, we are able to take $X_0 = i$ and $Y_0$ to be a random variable with distribution $\pi$, and then by (2) we will have

$$|p_{ij}^{(n)} - \pi_j| \to 0, \quad \text{as } n \to \infty.$$

2. Claim: This bivariate Markov chain is irreducible.

Proof of Claim:

We want to take any states $(i_x, i_y)$ and $(j_x, j_y)$ and find an integer $l$ such that

$$p_{(i_x,i_y),(j_x,j_y)}^{(l)} > 0.$$

• Take any states $(i_x, i_y)$ and $(j_x, j_y)$ in $S^2$.

• Since the original chain is irreducible, $i_x \leftrightarrow j_x$ and $i_y \leftrightarrow j_y$. ie: There exist integers $n$ and $m$ such that $p_{i_x,j_x}^{(n)} > 0$ and $p_{i_y,j_y}^{(m)} > 0$. (There are also two more integers that reverse these transitions, but we don’t care about them.)
Proof of Theorem $\pi_2$ Continued:

- Since the original chain is aperiodic, states $j_x$ and $j_y$ have period 1. Hence, by the claim preceding this Theorem, there exists a $K$ such that
  \[ p_{j_x,j_x}^{(k+K)} > 0 \quad \text{and} \quad p_{j_y,j_y}^{(k+K)} > 0 \]
  for all $k$.

Specifically,
  \[ p_{j_x,j_x}^{(m+K)} > 0 \quad \text{and} \quad p_{j_y,j_y}^{(n+K)} > 0. \]

- Therefore,
  \[ p_{(i_x,i_y),(j_x,j_y)}^{(n+m+K)} > 0 \]
  since the components move independently.

3. Since the two coordinates are independent,

\[ \pi_{(i_x,i_y)} = \pi_{i_x} \cdot \pi_{i_y} \]

defines a stationary distribution for the Markov chain.

Proof:

We need to show that

\[ \pi_{(j_x,j_y)} = \sum_{i_x,i_y} \pi_{(i_x,i_y)}p_{(i_x,i_y),(j_x,j_y)} \]

Well,

\[
\sum_{i_x,i_y} \pi_{(i_x,i_y)}p_{(i_x,i_y),(j_x,j_y)} = \sum_{i_x,i_y} \pi_{(i_x,i_y)}p_{(i_x,i_y),(j_x,j_y)}
= \sum_{i_x} \sum_{i_y} \pi_{i_x} \pi_{i_y}p_{i_x,j_x}p_{i_y,j_y}
= \sum_{i_x} \pi_{i_x}p_{i_x,j_x} \sum_{i_y} \pi_{i_y}p_{i_y,j_y}
= \sum_{i_x} \pi_{i_x}p_{i_x,j_x} \pi_{j_y} \quad (\pi \ \text{stationary})
= \pi_{j_y} \sum_{i_x} \pi_{i_x}p_{i_x,j_x}
= \pi_{j_y} \pi_{j_x} \overset{\text{def}}{=} \pi_{(j_x,j_y)} \quad (\pi \ \text{stationary})
\]
Proof of Theorem $\pi 2$ Continued:

4. Since $\pi(i_x, i_y)$ is stationary for the bivariate chain, if we can show that $\pi(i_x, i_y) > 0$ for all states $(i_x, i_y)$, we will have, by Lemma ($\pi 1$), that all states in the bivariate chain are recurrent.

Proof of $\pi(i_x, i_y) > 0$:

- $$\pi(i_x, i_y) = \pi_i x \cdot \pi_i y$$
  
  So, we need to show that $\pi_i x > 0$ and $\pi_i y > 0$ for all states $i_x, i_y$ in $S$.

  ie: We need to show $\pi_j > 0$ for all $j \in S$.

- Since $\pi$ is a distribution, $\sum_i \pi_i = 1$ implies that there is at least one state $i^*$ such that $\pi_{i^*} > 0$.

  Since $\pi$ is a stationary distribution, we have that

  $$\pi_j = \sum_i \pi_i p_{i,j}^{(n)}$$

  for any fixed time point $n$.

- Choose $n$ so that $p_{i^*,j}^{(n)} > 0$. We can do this since the original Markov chain is irreducible.

- Then we have that

  $$\pi_j = \sum_i \pi_i p_{i,j}^{(n)} \geq \pi_{i^*} p_{i^*,j}^{(n)} > 0.$$  

5. Let $T = \min\{n \geq 0 : X_n = Y_n\}$ and let $T(x) = \min\{n \geq 0 : X_n = Y_n = x\}$.

  Since the bivariate chain $\{(X_n, Y_n)\}$ is irreducible (can get to $(x, x)$) and recurrent (will get to $(x, x)$), we have that

  $$T(x) < \infty \quad \text{with probability 1.}$$

  So,

  $$T < T(x) < \infty \quad \text{with probability 1.}$$

6. Claim: $P(X_n = j, T \leq n) = P(Y_n = j, T \leq n)$.

   (“On $\{T \leq n\}$, $X_n$ and $Y_n$ have the same distribution.”)
Proof of Theorem $\pi 2$ Continued:

Proof:

\[
P(X_n = j, T \leq n) = \sum_{u=0}^{n} P(X_n = j, T = u)
\]

\[
= \sum_{u=0}^{n} \sum_{i} P(X_n = j, X_u = i, T = u)
\]

\[
= \sum_{u=0}^{n} \sum_{i} P(X_n = j | X_u = i, T = u) \cdot P(X_u = i, T = u)
\]

\[
\stackrel{M.P.}{=} \sum_{u=0}^{n} \sum_{i} P(X_n = j | X_u = i) \cdot P(X_u = i, T = u)
\]

\[
= \sum_{u=0}^{n} \sum_{i} P(Y_n = j | Y_u = i) \cdot P(Y_u = i, T = u)
\]

\[
= P(Y_n = j, T \leq n)
\]

In the second to last equality, the first factor came from the fact that \{X_n\} and \{Y_n\} have the same transition law. The second factor came from the fact that at time $T$, $X_n = Y_n$.

7. Note that

\[
P(X_n = j) = P(X_n = j, T \leq n) + P(X_n = j, T > n)
\]

\[
\overset{\text{Step 6}}{=} P(Y_n = j, T \leq n) + P(X_n = j, T > n)
\]

\[
\leq P(Y_n = j) + P(X_n = j, T > n).
\]

Similarly, we have that

\[
P(Y_n = j) \leq P(X_n = j) + P(X_n = j, T > n).
\]

So,

\[
|P(X_n = j) - P(Y_n = j)| \leq P(X_n = j, T > n) + P(Y_n = j, T > n)
\]

8. Summing over $j$, we get

\[
\sum_j |P(X_n = j) - P(Y_n = j)| \leq 2 \cdot P(T > n)
\]

regardless of the initial values for \{X_n\} and \{Y_n\}. 

Proof of Theorem $\pi2$ Continued:

9. Therefore, if we let $X_0 = i$ and $Y_0 \sim \pi$, we get

$$\sum_j |p_{i,j}^{(n)} - \pi_j| \leq 2 \cdot P(T > n) \to 0 \quad \text{as } n \to \infty$$

since $T < \infty$ with probability 1.

Therefore

$$p_{i,j}^n \to \pi_j \quad \text{as } n \to \infty$$

for all $i, j \in S$. □
Proof of Theorem π3:

Proof of Existence:

• Fix a state $k$.

• Let $N_i$ be the number of visits to state $i$ between two consecutive visits to state $k$.
  (In the case that $k = i$, count the last visit to $k$ but not the first. Then we have $N_k = 1$.)

• Let $T_k = \min\{n \geq 1 : X_n = k\}$. Note that
  $$N_i = \sum_{n=1}^{T_k} I\{X_n=i\} = \sum_{n=1}^{\infty} I\{X_n=i, T_k \geq n\}.$$ 

• Define $a_k(i) = \mathbb{E}_k[N_i]$. Then
  $$a_k(i) = \mathbb{E}_k[N_i] = \mathbb{E}_k[\sum_{n=1}^{\infty} I\{X_n=i, T_k \geq n\}]$$
  $$= \sum_{n=1}^{\infty} \mathbb{E}_k[I\{X_n=i, T_k \geq n\}]$$
  $$= \sum_{n=1}^{\infty} P_k(X_n = i, T_k \geq n)$$
  $$= \sum_{n=1}^{\infty} P(X_n = i, T_k \geq n|X_0 = k).$$

• Let $S$ be the state space for the Markov chain. Note that
  $$\sum_{i \in S} a_k(i) = \sum_{i \in S} \sum_{n=1}^{\infty} P(X_n = i, T_k \geq n|X_0 = k)$$
  $$= \sum_{n=1}^{\infty} P(T_k \geq n|X_0 = k)$$
  $$= \mathbb{E}_k[T_k]$$

Since the chain is positive recurrent, this expectation is finite. Let’s call it $m_k$.

• We can now define a probability distribution over $S$ with
  $$p_i := \frac{a_k(i)}{\sum_{j \in S} a_k(j)} = \frac{a_k(i)}{m_k}$$
  for all $i \in S$. 
Proof of Theorem $\pi 3$, Existence Continued:

- Note that
  
  $P(X_n = i, T_k \geq n|X_0 = k) = \sum_{j \in S} P(X_n = i, X_{n-1} = j, T_k \geq n|X_0 = k)$.

Also note that the term $P(X_n = i, X_{n-1} = j, T_k \geq n|X_0 = k)$ is zero when $j = k$. Thus, we have

$P(X_n = i, T_k \geq n|X_0 = k) = \sum_{j \neq k} P(X_n = i, X_{n-1} = j, T_k \geq n|X_0 = k)$

$= \sum_{j \neq k} P(X_n = i|X_{n-1} = j, T_k \geq n, X_0 = k) \cdot P(X_{n-1} = j, T_k \geq n|X_0 = k)$

- Since $T_k \geq n$, $X_n$ could be $k$ or could be anything else. It is not restricted and we have

$P(X_n = i|X_{n-1} = j, T_k \geq n, X_0 = k) = p_{ji}$,

the usual transition probability for the Markov chain.(Note that this would not be the case if, say $T_k \geq n$ were replaced by $T_k = n$ or $T_k > n$.)

Thus, we have

$P(X_n = i, T_k \geq n|X_0 = k) = \sum_{j \neq k} p_{ji} P(X_{n-1} = j, T_k \geq n|X_0 = k)$

$= \sum_{j \neq k} p_{ji} P(X_{n-1} = j, T_k \geq n - 1|X_0 = k)$

since $j \neq k$.

- Note that $P(X_1 = i, T_k \geq 1|X_0 = k) = P(X_1 = i|X_0 = k) = p_{ki}$. So, we have

$a_k(i) = \sum_{n=1}^{\infty} P(X_n = i, T_k \geq n|X_0 = k)$

$= p_{ki} + \sum_{n=2}^{\infty} P(X_n = i, T_k \geq n|X_0 = k)$

$= p_{ki} + \sum_{n=2}^{\infty} \sum_{j \neq k} p_{ji} P(X_{n-1} = j, T_k \geq n - 1|X_0 = k)$
Proof of Theorem $\pi 3$, Existence Continued:

- Interchanging the order of summation, we have

$$a_k(i) = p_{ki} + \sum_{j \neq k} p_{ji} \sum_{n=2}^{\infty} P(X_{n-1} = j, T_k \geq n - 1 | X_0 = k)$$

$$= p_{ki} + \sum_{j \neq k} p_{ji} \sum_{n=1}^{\infty} P(X_n = j, T_k \geq n | X_0 = k)$$

$$= p_{ki} + \sum_{j \neq k} p_{ji} a_k(i)$$

$$= \sum_{j \in S} p_{ji} a_k(i)$$

since $a_k(k) = E_k[N_k] = E_k[1] = 1$.

- We have shown that

$$a_k(i) \sum_{j \in S} p_{ji} a_k(i).$$

Dividing both sides by $m_k$ gives us

$$p_i \sum_{j \in S} p_{ji} p_j.$$ 

This tells us that the probability distribution $\{p_i : i \in S\}$ is stationary for this Markov chain!

(End of existence part of proof.)

The rest of the proof is coming soon!