

Piston Dispersive Shock Wave Problem

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The piston shock problem is a classical result of shock wave theory. In this work, the analogous dispersive shock wave (DSW) problem for a fluid described by the nonlinear Schrödinger equation is analyzed. Asymptotic solutions are calculated for a piston (step potential) moving with uniform speed into a dispersive fluid at rest. In contrast to the classical case, there is a bifurcation of shock behavior where, for large enough piston velocities, the DSW develops a periodic wave train in its wake with vacuum points and a maximum density that remains fixed as the piston velocity is increased further. These results have application to Bose-Einstein condensates and nonlinear optics.

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The study of dispersive shock waves (DSWs) has gained significance due to the recent experimental realization of DSWs in a Bose-Einstein condensate (BEC) [1,2] and the propagation of light through a nonlinear defocusing medium [3]. Comparisons between classical, viscous shock waves (VSWs), and DSWs have been discussed in the context of single shocks [2] and the interaction of two shocks [4], revealing appealing similarities but also important differences. Motivated by the classical VSW piston problem, we consider the generation of a DSW by a piston moving into a dispersive fluid at rest.

The piston shock problem is one of the canonical problems in the theory of VSWs. A uniform gas is held at rest in a long, cylindrical chamber with a piston at one end. When the piston is impulsively moved into the gas with constant speed, a region of higher density builds between the piston and a shock front that propagates ahead of it. An elegant asymptotic (zero dissipation limit) solution to this problem is well known and relates the shock speed to the speed of the piston and the initial density of the gas (see, e.g., [5] and the discussion below).

In this work, we consider the analogous problem of a “piston” moving with constant speed into a steady, dispersive fluid: e.g., a BEC or nonlinear optics. The piston in this case is a step potential that moves with uniform speed. This potential could be realized in a BEC with a repulsive dipole beam and in nonlinear optics with a local change in the index of refraction. One expects, in analogy with the classical, viscous case, the generation of a dispersive shock wave. As we will show, this is indeed the case. However, in contrast to the viscous case, there are *two* types of asymptotic behavior, depending on the piston speed. For smaller piston velocities, a region of larger density builds between the piston and a DSW. For large enough piston velocities, a locally periodic wave train is generated between the piston and the DSW that has no VSW correlate. This wave train oscillates between the vacuum state (zero density) and a

maximum density that is independent of further increase in the piston velocity.

DSWs can be studied using the Whitham averaging method [6]. This technique has been successfully applied to many DSW problems including collisionless shocks in plasma [7], undular bores in hydrodynamics [8], Bose-Einstein condensates [2,9,10], fiber optics [11], the generation of ultrashort lasers [12], and DSW interactions [4]. A related class of moving boundary shock problems was studied as an asymptotic reduction of two-dimensional, steady, supersonic flow of a dispersive fluid around an obstacle [13].

We consider the one-dimensional (1D) nonlinear Schrödinger equation (NLS) with a potential [also known as the Gross-Pitaevskii (GP) equation]

$$i\varepsilon\Psi_t = -\frac{\varepsilon^2}{2}\Psi_{xx} + V_0(x,t)\Psi + |\Psi|^2\Psi, \quad 0 < \varepsilon \ll 1. \quad (1)$$

This equation models the mean field of a quasi-1D BEC [14] and the slowly varying envelope of the electromagnetic field propagating through a Kerr medium [15]. The small parameter ε is inversely proportional to the number of atoms in the BEC [2] or, after rescaling, inversely proportional to the maximum initial intensity of the electromagnetic field. For all calculations in this work, we assume $\varepsilon = 0.015$, a typical experimental value for BEC [2]. The piston problem is modeled with a temporally and spatially varying step potential $V_0(x,t) = V_{\max}H(v_p t - x)$, where $H(y)$ is the Heaviside step function. The piston strength and speed are V_{\max} and v_p , respectively. The initial conditions are $\Psi(x,0) \rightarrow \sqrt{\rho_R}$ as $x \rightarrow \infty$, $\Psi(x,0) \rightarrow 0$ as $x \rightarrow -\infty$. Because the strength of the piston is large, $V_{\max} \gg \rho_R$, the density or intensity rapidly decays to zero near the origin. We assume that the wave function Ψ is in the “ground state” of the step potential $V_{\max}H(-x)$ when $t \leq 0$. For all calculations in this work, $\rho_R = 0.133$.

It is useful to view Eq. (1) in its hydrodynamic form by making the transformation $\Psi = \sqrt{\rho} \exp[\frac{i}{\varepsilon} \int_0^x u(x', t) dx']$ and inserting this expression into the first two local conservation equations for the GP equation,

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + \left(\rho u^2 + \frac{1}{2}\rho^2\right)_x &= \frac{\varepsilon^2}{4}[\rho(\log\rho)_{xx}]_x - \rho V_{0,x}, \end{aligned} \quad (2)$$

where ρ is the dispersive fluid “density” and u is the dispersive fluid “velocity”. These equations are similar to the Navier-Stokes and shallow water equations of fluid dynamics except that the viscous terms have been replaced by the dispersive term with coefficient $\varepsilon^2/4$.

To motivate the discussion of the DSW piston problem, we briefly consider the analogous VSW piston problem in shallow water (see, e.g., [5]). The 1D equations are equivalent to Eqs. (2) when $\varepsilon = 0$, $V \equiv 0$, and a dissipative regularization is used whenever a shock forms. The asymptotic solution is found by assuming a simple wave (pure VSW) and the boundary condition

$$u(v_p t, t) \equiv u_L = v_p \quad (3)$$

at the piston. This condition is derived from the continuity equation in (2) and is therefore applicable to the dispersive case also. The shock speed (v_s) is always larger than the piston speed; i.e., one finds $v_s - v_p = v_p \rho_R / (\rho_L - \rho_R) > 0$ where $\rho_L > \rho_R$ is the fluid density between the piston and the shock. As we now show, the DSW piston problem admits quite different behavior. Note that the asymptotic solutions for the VSW and DSW piston problems are the *same* when the piston is retracted, $v_p < 0$, because no shock waves develop.

We convert the piston DSW problem into a moving boundary value problem where appropriate boundary conditions are imposed at the piston front. First we solve the piston DSW problem for sufficiently small positive piston velocities v_p . The first boundary condition at the piston for the local fluid velocity is Eq. (3). In addition, we require a boundary condition for the density.

The Whitham theory of DSWs involves a system of quasilinear, first order, hyperbolic equations [6]. These Whitham equations describe the slow evolution of a periodic wave’s parameters. The simplest nontrivial solutions to these equations are known as *simple waves*, where only one dependent variable varies in space and time and the rest are constant (a pure DSW or rarefaction wave). In analogy with viscous fluid dynamics, we assume a simple wave DSW solution, but in this case to the Whitham equations. This determines a density ρ_L at the piston. In order to connect to the uniform state ahead of the piston $\rho_R < \rho_L$, we must have a single DSW for v_p sufficiently small ($v_p < 2\sqrt{\rho_R}$). As we will show below, a “vacuum state” is created when $v_p \geq 2\sqrt{\rho_R}$, and we find a uniform traveling wave (TW) with speed v_p , instead of the constant

density ρ_L , adjacent to the DSW. Now we derive the asymptotic piston DSW.

At the time $t = 0^+$, we assume that there is a discontinuity in the fluid variables due to the impulsive motion of the piston at $t = 0$:

$$\begin{aligned} \rho(x, 0^+) &= \begin{cases} \rho_L, & x = 0 \\ \rho_R, & x > 0 \end{cases}, \\ u(x, 0^+) &= \begin{cases} u_L = v_p, & x = 0 \\ u_R = 0, & x > 0 \end{cases}. \end{aligned} \quad (4)$$

This discontinuity is regularized by a slowly modulated traveling wave solution to Eq. (2) with $V(x, t) \equiv 0$ [9]:

$$\begin{aligned} \rho(x, t, \theta) &= \lambda_3 - [\lambda_3 - \lambda_1] dn^2(\theta; m), \quad m = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}, \\ u(x, t, \theta) &= V - \sigma \frac{\sqrt{\lambda_1 \lambda_2 \lambda_3}}{\rho(x, t, \theta)}, \quad 0 < \lambda_1 < \lambda_2 < \lambda_3, \\ \sigma &= \pm 1, \quad \frac{\partial \theta}{\partial x} = \sqrt{\lambda_3 - \lambda_1} / \varepsilon, \\ \frac{\partial \theta}{\partial t} &= -V \sqrt{\lambda_3 - \lambda_1} / \varepsilon, \end{aligned} \quad (5)$$

where the parameters λ_i , $i = 1, 2, 3$, and V satisfy

$$\begin{aligned} \lambda_1 &= \frac{1}{16}(r_1 - r_2 - r_3 + r_4)^2, \\ \lambda_2 &= \frac{1}{16}(-r_1 + r_2 - r_3 + r_4)^2, \\ \lambda_3 &= \frac{1}{16}(-r_1 - r_2 + r_3 + r_4)^2, \\ V &= \frac{1}{4}(r_1 + r_2 + r_3 + r_4), \end{aligned}$$

and the Riemann invariants, $r_i(x, t)$, evolve according to the Whitham equations,

$$\frac{\partial r_i}{\partial t} + v_i(r_1, r_2, r_3, r_4) \frac{\partial r_i}{\partial x} = 0, \quad i = 1, 2, 3, 4.$$

The velocities v_i can be expressed in terms of $K(m)$ and $E(m)$, complete elliptic integrals of the first and second kind, respectively [9]. In this work, we only require knowledge of $v_3 = V - \frac{1}{2}(r_4 - r_3)[1 - \frac{(r_4 - r_2)E(m)}{(r_3 - r_2)K(m)}]^{-1}$.

In order to find a simple wave solution to the Whitham equations, we require that only one of the parameters r_i spatially varies and that the initial data for all the parameters r_i properly characterize the initial data in Eq. (4) with the spatial average of Eq. (5). We use the method of initial data regularization [2,4,11,12] to find

$$\begin{aligned} r_1 &\equiv -2\sqrt{\rho_R}, \quad r_2 \equiv 2\sqrt{\rho_R}, \quad r_4 \equiv 2v_p + 2\sqrt{\rho_R}, \\ r_3(x, 0^+) &= \begin{cases} 2\sqrt{\rho_R}, & x = 0 \\ 2v_p + 2\sqrt{\rho_R}, & x > 0 \end{cases}, \quad \sigma \equiv 1, \\ \rho(v_p t, t) &\equiv \rho_L = \left(\frac{1}{2}v_p + \sqrt{\rho_R}\right)^2. \end{aligned} \quad (6)$$

The last equation, the boundary condition for the density at the piston, comes from the simple wave assumption. A

self-similar, rarefaction solution for r_3 is found giving rise to a pure DSW propagating ahead of the piston with trailing and leading edge speeds, respectively [2,9]:

$$v_s^- = \frac{1}{2}v_p + \sqrt{\rho_R}, \quad v_s^+ = \frac{2v_p^2 + 4v_p\sqrt{\rho_R} + \rho_R}{v_p + \sqrt{\rho_R}}. \quad (7)$$

Figure 1, left, depicts the asymptotic piston DSW solution for a small piston velocity. The minimum values of the density and velocity occur at the trailing edge of the DSW and are [2,9]:

$$\rho_{\min} = \left(\sqrt{\rho_R} - \frac{1}{2}v_p\right)^2, \quad u_{\min} = -v_p \frac{\left(\sqrt{\rho_R} + \frac{1}{2}v_p\right)}{\left(\sqrt{\rho_R} - \frac{1}{2}v_p\right)}. \quad (8)$$

The maximum values occur between the piston and the DSW: $\rho_{\max} = \rho_L = (v_p/2 + \sqrt{\rho_R})^2$, $u_{\max} = u_L = v_p$.

The piston velocity can be greater than the trailing DSW velocity calculated by use of Eq. (7), $v_p \geq v_s^-$ if $v_p \geq 2\sqrt{\rho_R}$. When $v_p = 2\sqrt{\rho_R}$, ρ vanishes at the piston (there is a so-called vacuum point [2,9]) and a modification of the solution is required. To find a simple wave solution for large piston velocities, we must derive new conditions for the parameters r_i . We modify the DSW solution by introducing a locally periodic traveling wave (TW) between the piston and the trailing edge of the DSW. The *vacuum condition*, $\rho = 0$, is satisfied in Eq. (5) when

$$\lambda_1 = 0 \Rightarrow r_1 - r_2 - r_3 + r_4 = 0 \Rightarrow r_4 - r_3 = 4\sqrt{\rho_R}. \quad (9)$$

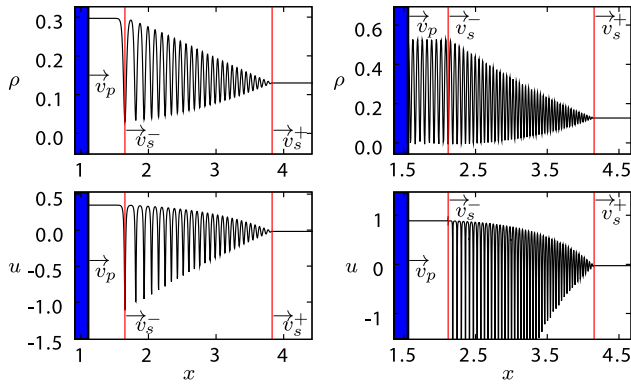


FIG. 1 (color online). Left: asymptotic piston DSW solution at time $t = 3$ for $v_p = \sqrt{\rho_R} = 0.365$. The vertical lines mark the left and right edges of the DSW moving with speeds $v_s^- = 0.548$ and $v_s^+ = 1.278$, respectively. Right: asymptotic piston DSW solution at time $t = 1.7$ when $v_p = 2.5\sqrt{\rho_R} = 0.912$. A locally periodic region connects the piston to the trailing edge of the DSW. The density minima in this region are zero and the velocity is theoretically undefined (infinite) at these points. The density maxima are $4\rho_R = 0.532$. DSW speeds are $v_s^- = 1.245$, $v_s^+ = 2.452$.

Note that the fluid velocity is undefined at a vacuum point, even though the vacuum points have a well-defined propagation speed through the fluid. We assume that condition (9) holds for $v_p > 2\sqrt{\rho_R}$ as well. One more condition is required to completely determine r_4 and $r_3(x = 0, t = 0^+)$; r_1, r_2 , and $r_3(x > 0, 0^+)$ are determined by the initial data (4). Because there is a locally periodic TW between the piston and DSW, we assume that the velocity of the TW equals the piston velocity. This is the *TW velocity condition*:

$$V = v_p \Rightarrow r_1 + r_2 + r_3 + r_4 = 4v_p \Rightarrow r_3 + r_4 = 4v_p. \quad (10)$$

Given the initial data in Eq. (4) and the two conditions (9) and (10) only r_3 and σ in the initial data of Eq. (6) are altered:

$$r_3(x, 0^+) = 2v_p \mp 2\sqrt{\rho_R}, \quad \sigma(x, 0^+) = \mp 1, \quad x \gtrless 0. \quad (11)$$

Note that $\sigma = -1$ in the locally periodic region. The motivation for introducing a TW comes from the Whitham analysis where, to satisfy the simple wave condition for $v_p > 4\sqrt{\rho_R}$, $r_3(0, 0^+)$ is taken to be larger than $r_2(0, 0^+)$ [compare Eq. (11) with Eq. (6)].

Figure 1, right, depicts the asymptotic DSW solution for $v_p > 2\sqrt{\rho_R}$. Several properties of this DSW solution are worth noting. The density between the piston and the DSW oscillates between the values

$$\rho_{\min} = 0 \quad \text{and} \quad \rho_{\max} = 4\rho_R, \quad (12)$$

independent of the piston velocity v_p , while the TW in this region propagates with the velocity $V = v_p$. Note that since the vacuum condition (9) holds everywhere inside

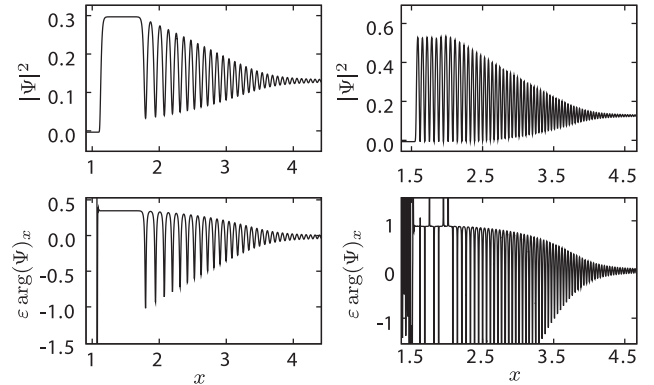


FIG. 2. Numerical solutions to Eq. (1) for the piston problem. Left: the density (upper) and velocity (lower) for the same parameters as in Fig. 1, left; numerically calculated trailing edge speed is 0.557, approximately the theoretical value 0.548. Right: the density (upper) and velocity (lower) for the same parameters as in Fig. 1, right; numerically calculated TW velocity of locally periodic wave is 0.912, approximately the same as the piston speed 0.913.

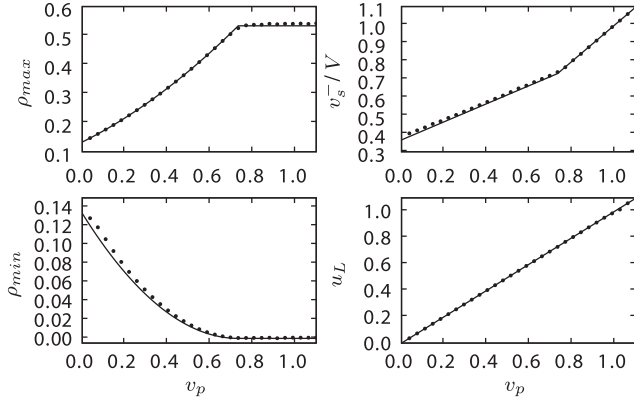


FIG. 3. Comparisons of the analytical theory (solid curves) and numerical simulations (\bullet) for different DSW parameters as functions of the piston velocity v_p . Left: ρ_{\max} (top) and ρ_{\min} (bottom). Maximum absolute error in ρ_{\max} from Eqs. (6) and (12) is 0.0088. Maximum absolute error in ρ_{\min} from Eq. (8) is 0.010. Right: top depicts the speed of the trailing edge of the piston DSW for $v_p < 2\sqrt{\rho_R} = 0.73$ and TW velocity V of locally periodic wave for $v_p \geq 2\sqrt{\rho_R}$. Bottom shows the validity of the boundary condition (3). Maximum absolute error in v_s^-/V from Eqs. (7) and (10) is 0.018. Maximum absolute error in u_L from Eq. (3) is 0.013.

the TW trailing the DSW, the velocity in this region, from Eq. (5), is $u = V = v_p$ everywhere (except at the vacuum points, where the velocity is undefined). The wavelength of the TW is $l = 2\epsilon K(4\rho_R/v_p^2)/v_p$. The DSW propagates with trailing edge speed (also the propagation speed of the rightmost vacuum point where σ changes sign) $v_s^- = v_p + (v_p + 3\sqrt{\rho_R})\left[\frac{v_p E(4\rho_R/v_p^2)}{(v_p - 2\sqrt{\rho_R})K(4\rho_R/v_p^2)} - 1\right]^{-1}$, and leading edge speed v_s^+ , the same as that given in Eq. (7). The number of vacuum points increases linearly with time: $N_{\text{vac}}(t) \approx \left\lfloor \frac{v_s^- - v_p}{l} t \right\rfloor = \left\lfloor \frac{(v_s^- - v_p)v_p}{2\epsilon K(4\rho_R/v_p^2)} t \right\rfloor$.

We perform direct numerical simulations of Eq. (1) to verify the assumptions we have made, such as the boundary conditions (3) and (6), the vacuum and TW velocity conditions (9) and (10), and the trailing edge DSW speed v_s^- of Eq. (7). All of our assumptions agree well with the numerical simulations shown in Figs. 2 and 3.

Numerically calculated piston DSWs for both moderate ($v_p = \sqrt{\rho_R}$) and large ($v_p = 2.5\sqrt{\rho_R}$) piston velocities are shown in Fig. 2. For the slower piston velocity in Fig. 2, left, the solution is similar to the asymptotic result in Fig. 1, left. The piston DSW corresponding to the large piston velocity in Fig. 2, right, is very similar to the asymptotic result in Fig. 1, right. The vacuum condition in Eq. (9) predicts $u = v_p$ everywhere in the trailing wave region,

except at vacuum points, where it is undefined. This is reflected in the numerical calculation as very large spikes in the velocity when the density approaches zero.

In conclusion, we have solved the dispersive shock wave piston problem for arbitrary positive piston velocities in systems described by the nonlinear Schrödinger equation, demonstrating the existence of a bifurcation in shock behavior. For small piston velocities, a DSW propagates ahead of the piston, as in the viscous shock piston problem. For large enough piston velocities, a wave train oscillating between the vacuum state and a saturated maximum density propagates between the piston and DSW, behavior unique to DSWs.

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