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Emergence of synchronization in complex networks of interacting dynamical systems

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Abstract

We study the emergence of coherence in large complex networks of interacting heterogeneous dynamical systems. We show that for a large class of dynamical systems and network topologies there is a critical coupling strength at which the systems undergo a transition from incoherent to coherent behavior. We find that the critical coupling strength at which this transition takes place is $k_c = (Z\lambda)^{-1}$, where Z depends only on the uncoupled dynamics of the individual systems on each node, while λ is the largest eigenvalue of the network adjacency matrix. Thus we achieve a separation of the problem into two parts, one depending solely on the node dynamics, and one depending solely on network topology.
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1. Introduction

There has recently been much interest in the properties of large complex networks [1,2]. Another concern has been the study of dynamical processes taking place on such networks and of how network structure impacts the dynamics. One of the most remarkable phenomena in the study of coupled dynamical systems is their spontaneous synchronization. Under appropriate conditions, a collection of interacting dynamical systems with possibly different parameters can achieve a state of macroscopic coherence. This phenomenon is extremely important in several fields, ranging from biology to engineering [3,4]. The interactions among the different dynamical systems often constitute a large complex network, and it is thus important to study the effect of network structure on the onset of collective synchronization.

When studying dynamical systems coupled by a network of interactions, there are two important aspects of the problem that can be changed independently (and, as we shall see, have an independent effect): the dynamics of the individual, uncoupled dynamical systems, and the topology of the interaction network. Models of coupled dynamical systems have until very recently simplified the dynamics of the systems, the topology of their interactions, or both. The case of an all-to-all network of sine coupled phase oscillators corresponds to the well-known Kuramoto model [5]. This model considers N oscillators, each of which is described by a phase θ_j and has a frequency ω_j. The frequencies are assumed to be randomly drawn from a probability distribution $\Omega(\omega)$ independently of j, and $\Omega(\omega)$ is assumed to be symmetric around a single local maximum located (without loss of generality) at $\omega = 0$. Sinusoidal coupling is assumed so that θ_j evolves as $\dot{\theta}_j = \omega_j + k \sum_{m=1}^{N} \sin(\theta_m - \theta_j)$. In the limit $N \rightarrow \infty$, for coupling strengths k less than a critical coupling strength k_c, the phases of the oscillators are incoherent, i.e., θ_j are uniformly distributed on $[0, 2\pi)$. For values of the coupling strength k larger than k_c, a significant fraction of the oscillators evolve with a common frequency. The value of the critical coupling strength k_c depends on the frequency distribution Ω. The Kuramoto model has become a classical paradigm for the

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emergence of synchronization in an ensemble of heterogeneous oscillators (see [6] for reviews).

The effect of more general dynamics or a more general interaction network has been recently studied, and it has been found that in both cases there is still a transition to coherent behavior. In particular, for the Kuramoto phase oscillator model generalized to the case of a general interaction network described by an adjacency matrix [7–13], we [7,8] found that there is still a transition to synchrony at a critical coupling strength that depends separately on the largest eigenvalue of the adjacency matrix and the frequency distribution. Also, for equal coupling strength, all-to-all coupled collections of many dynamical systems with more general dynamics (e.g., mixed collections of chaotic and periodic oscillators, chaotic maps, etc.) [14–18], there is also a transition to synchrony at coupling strengths that depend on the dynamics of the uncoupled systems. In the case of more general dynamics, by a transition to synchrony we mean that an average over the states of all the individual dynamical systems displays coherent oscillation. In particular, while the time series of the state of any one of the systems may appear chaotic, on averaging, the temporally chaotic variations cancel, and a coherent periodic oscillation of the globally averaged state is revealed.

Recently, we presented a brief report showing how to generalize these efforts to the case of general dynamics and a general interaction network for interacting maps [19]. Our main result was that the critical coupling strength for the onset of synchronization depends separately on the largest eigenvalue of the adjacency matrix of the network, and on the dynamics of the individual, uncoupled oscillators. We present here a more detailed analysis of our previous results and an extension to collections of continuous time oscillators (including periodic, chaotic, or mixed ensembles).

This paper is organized as follows. In Section 2 we review the results for general dynamics in an all-to-all network and for the Kuramoto model generalized to complex networks. In Section 3 we present our theory for general dynamical systems coupled in a network. In Section 4 we present numerical examples, and in Section 5 we present our conclusions.

2. Background

In Section 2.1 we review previous results for the onset of synchronization in the case of globally coupled dynamical systems [17,18], while in Section 2.2 we review results for the onset of synchronization in collections of phase oscillators coupled in a network [7,8].

2.1. Globally coupled maps

Reference [17] investigates systems of globally coupled maps defined by

$$x_{n+1}^{(j)} = f(x_n^{(j)}, \mu_j) + w_n^{(j)} + kg(x_n^{(j)})[\overline{q(x_n)} - \langle q \rangle]. \quad (1)$$

Here $j = 1, 2, \ldots, N$ labels the map, and we are interested in large N; $f(x_n^{(j)}, \mu_j)$ determines the uncoupled dynamics of each individual map with a parameter vector μ_j; for each map j

the parameter vector μ_j is chosen randomly and independently of j with a probability distribution $p(\mu)$; and the term $w_n^{(j)}$ is a random noise which is assumed to be statistically independent of j and n and to satisfy $E[w_n^{(j)}] = 0, E[w_n^{(j)} w_m^{(l)}] = \sigma^2 \delta_{nm} \delta_{jl}$, where $E[\cdot]$ represents the expected value. In the coupling term, k is a global coupling strength and the functions g and q are assumed to be smooth and bounded. The notation $\langle \cdot \rangle$ represents the average over the distribution of the parameter vector μ_j and over the natural measures of the attractors of the noisy uncoupled ($k = 0$) system. More precisely, if an individual noisy uncoupled system with parameter vector μ has a unique attractor with a natural measure $m_\mu(x)$, then

$$\langle q \rangle = \iint q(x) \mathrm{d}m_\mu(x) p(\mu) \mathrm{d}\mu. \quad (2)$$

Alternatively, $\langle q \rangle$ is the μ-average of the infinite time average of $q(x_n)$ over a typical orbit x_n of the noisy uncoupled system. We remark that $\langle q \rangle$ is independent of time. The notation $\overline{q(x_n)}$ indicates an average over j,

$$\overline{q(x_n)} = \frac{1}{N} \sum_{j=1}^{N} q(x_n^{(j)}). \quad (3)$$

We note that $\overline{q(x_n)}$, in general, depends on the time n. We also note that x can be a vector for the situation of multidimensional individual maps. For simplicity, in what follows x will be considered a scalar; thus f, g, and q are scalar functions.

Consider what happens if the initial conditions for system (1) are chosen distributed according to the natural measure of the attractors of the uncoupled systems [imagine that system (1) is left running for a long time with $k = 0$, and then k is suddenly changed to a nonzero value]. Because of the large number of terms in the sum (3) and the fact that x_n are distributed according to the measure of the uncoupled attractors, we have

$$\langle q \rangle \approx \overline{q(x_n)}, \quad (4)$$

and therefore the coupling term in Eq. (1) approximately (for large $N < \infty$) vanishes and the oscillators continue evolving independently of each other. We refer to this situation as the *incoherent state* [17], and, as we have just argued, this state is a solution of system (1) in the limit $N \to \infty$. The onset of synchronization can be associated with the linear instability of the incoherent state. By considering linear perturbations $\delta x_n^{(j)}$ from the incoherent state $x_n^{(j)}$, and perturbations $\delta_n^{(j)}$ of the uncoupled system

$$\delta_{n+1}^{(j)} = f'(x_n^{(j)}, \mu_j)\delta_n^{(j)}, \qquad \delta_0 = 1, \quad (5)$$

a dispersion relation determining the onset of instability of the incoherent state was obtained in Ref. [17]. Since this is a particular case of the theory presented in Section 3.1 we will present here the results and leave the details for the more general case.

Assuming that perturbations from the incoherent state grow exponentially in time as η^n, the dispersion relation obtained

in [17] is

$$1 = kZ(\eta), \tag{6}$$

where

$$Z(\eta) = \lim_{n \to \infty} \left\langle \sum_{p=0}^{n} \frac{q'(x_{p+1}) \delta_{n+1} g(x_p) \eta^{p-n-1}}{\delta_{p+1}} \right\rangle. \tag{7}$$

For large n, $\delta_{n+1}/\delta_{p+1}$ grows with n on average as $\chi(\mu, x_0)^{n-p}$, where $\chi(\mu, x_0)$ is the Lyapunov exponent for the uncoupled system with parameter vector μ and initial condition x_0. Thus, for the limit to converge we require

$$|\eta| > \chi \equiv \sup_{x_0, \mu} \chi(\mu, x_0). \tag{8}$$

Under this condition, we can interchange the sum and the average to obtain, after letting $m = n - p$ and $n \to \infty$,

$$Z(\eta) = \frac{1}{\eta} \sum_{m=0}^{\infty} \frac{Z_m}{\eta^m}, \tag{9}$$

where

$$Z_m = \left\langle \frac{\delta_{n+1}}{\delta_{n-m+1}} q'(x_{n+1}) g(x_{n-m}) \right\rangle. \tag{10}$$

The above is only valid, in principle, for $|\eta| > \chi$. For chaotic systems, $\chi > 1$, but in order to investigate the onset of coherence, we would like to study the case $|\eta| = 1$. In [17] it is argued that the Z_m decrease exponentially with m, so that $Z(\eta)$ defined as in Eq. (9) can be analytically continued to values $|\eta| < 1$.

The onset of instability corresponds to $|\eta| = 1$; thus setting $\eta = e^{i\omega}$ and solving

$$1 = kZ(e^{i\omega}), \tag{11}$$

one obtains values of k at which the incoherent state is marginally stable: from $\text{Im}(Z(e^{i\omega})) = 0$ one obtains critical frequencies and from these one can solve for k. The smallest positive solution k_+ and the largest negative solution k_- correspond to the onset of instability of the incoherent state as the coupling strength is increased or decreased from $k = 0$.

2.2. Networks of coupled phase oscillators

In Section 2.1 we reviewed previous results for *globally* coupled maps. Although the uncoupled dynamics considered are quite general, the coupling network is assumed to be all-to-all. In this section we review results for the case of simple dynamics (oscillators described only by their phase) on a complex network [7,8]. This is a direct generalization of the Kuramoto model described in the introduction to the case of more general connectivity of the oscillators. The system considered is

$$\dot{\theta}_j = \omega_j + k \sum_{m=1}^{N} A_{jm} \sin(\theta_m - \theta_j). \tag{12}$$

Here $j, m = 1, 2, \ldots, N$, ω_j is the frequency of oscillator j which is assumed to be randomly drawn from a probability

distribution $\Omega(\omega)$ independent of j. As in the Kuramoto model, the frequency distribution is assumed to be symmetric around a single local maximum located (without loss of generality) at $\omega = 0$.

The $N \times N$ matrix A whose elements appear in Eq. (12) determines the network of interactions: node m directly affects node j only if $A_{jm} \neq 0$. We refer to those nodes m for which $A_{jm} \neq 0$ as the *neighbors* of node j, and to $d_j = \sum_{m=1}^{N} A_{jm}$ as the *in-degree* of node j. The diagonal elements of A do not affect the dynamics specified by Eq. (12), and, for convenience, we will take them to be zero, $A_{nn} = 0$ (equivalently, we could include the constraint $m \neq j$ in the subsequent sums).

Defining a local order parameter by

$$r_j = \sum_{m=1}^{N} A_{jm} e^{i\theta_m}, \tag{13}$$

the macroscopic coherence in the collection of oscillators is quantified by the global order parameter

$$r = \frac{\sum_{j=1}^{N} r_j}{\sum_{j=1}^{N} d_j}. \tag{14}$$

Under the assumption that nodes have a large number of neighbors, a self consistent equation (the *time averaged theory*, TAT) was obtained in Ref. [7],

$$r_j = \sum_{|\omega_m| \leq kr_m} A_{jm} \sqrt{1 - \left(\frac{\omega_m}{kr_m}\right)^2}. \tag{15}$$

The incoherent state corresponds to $r_j = 0$ for all j, which is a solution of Eq. (15). Coherent macroscopic oscillations correspond to solutions of Eq. (15) with $r_j \neq 0$ for some set of j. In general, to find such states Eq. (15) must be solved numerically [a much quicker task than integrating Eq. (12)].

Assuming that each node's neighbors have a representative sample of frequencies, and averaging the individual terms in the summand of Eq. (15) over the frequencies, one obtains the approximation (*frequency distribution approximation*, FDA)

$$r_j = k \sum_{m=1}^{N} A_{jm} r_m \int_{-1}^{1} \Omega(zkr_m) \sqrt{1 - z^2} \, dz. \tag{16}$$

The onset of coherence corresponds to $r_j \to 0^+$, from which one obtains a critical coupling strength given by

$$k_c = \frac{2}{\pi \Omega(0)\lambda}, \tag{17}$$

where λ is the largest eigenvalue of the matrix A (the largest eigenvalue corresponds to the first mode becoming unstable as the coupling strength is increased from zero; the most negative eigenvalue should be used to find the onset of synchronization as the coupling strength is decreased from zero). For symmetric matrices or matrices with positive entries the eigenvalue with largest magnitude is real. For a large class of nonsymmetric

matrices with mixed positive/negative entries, we have found that if the average of the nonzero elements of the matrix is large enough, the eigenvalue with largest magnitude is also real and it is well separated from the eigenvalue with the next largest magnitude. For details, see the Appendix of Ref. [8].

3. Networks of coupled dynamical systems

In the previous section we have reviewed results in cases that simplified either the network topology (Section 2.1) or the dynamics at the nodes (Section 2.2). In this section we will consider the general case. In Section 3.1 we discuss networks of coupled discrete maps [19], while Section 3.2 considers networks of continuous time systems (ordinary differential equations).

3.1. Networks of coupled maps

Here we will consider networks of N coupled maps satisfying for $j = 1, 2, \ldots, N$,

$$
\begin{aligned}
x_{n+1}^{(j)} &= f(x_n^{(j)}, \mu_j) + w_n^{(j)} \\
&\quad + kg(x_n^{(j)}) \sum_{m=1}^{N} A_{jm}[q(x_n^{(m)}) - \langle q(x) \rangle].
\end{aligned}
\tag{18}
$$

Here, the notation is the same as in Section 2.1, with the addition of the matrix A, which was introduced in Section 2.2. We recall that, for simplicity, we assume the quantities in the previous equation to be scalar.

We are interested in studying system (18) for the case in which nodes have a large number of neighbors. In this case, if the initial conditions for system (18) are chosen distributed according to the natural measure of the attractors of the uncoupled systems, then, because of the large number of terms in the sum in the coupling term in Eq. (18), the fact that x_n are distributed according to the measure of the uncoupled attractors, and the lack of correlations between the parameter vectors and the network, we can approximate

$$
\sum_{m=1}^{N} A_{jm} q(x_n^{(m)}) \approx \langle q(x) \rangle \sum_{m=1}^{N} A_{jm}.
\tag{19}
$$

This is analogous to Eq. (3), but we have had to introduce the extra assumption of a large number of neighbors per node, as opposed to only requiring large N in the all-to-all case.

We see that in this more general case, under the previously mentioned assumption, the incoherent state is also (approximately) a solution of the system. Therefore, its linear stability can be studied using the same methods that were used in the all-to-all case. In the following, we will adapt these techniques to the more general case of system (18). Therefore, we will follow closely the ideas in Section 2.1 (or Ref. [17]), but adapted to the presence of the network.

We want to study the linear stability of the incoherent state. Thus, we assume that $x_n^{(i)}$ is in the incoherent state and introduce an infinitesimal perturbation $\delta x_n^{(i)}$. Linearization of

Eq. (18) produces

$$
\begin{aligned}
\delta x_{n+1}^{(j)} &= f'(x_n^{(j)}, \mu_j) \delta x_n^{(j)} \\
&\quad + kg(x_n^{(j)}) \sum_{m=1}^{N} A_{jm} q'(x_n^{(m)}) \delta x_n^{(m)}.
\end{aligned}
\tag{20}
$$

In order to solve Eq. (20) we consider (as in the variation of parameters method for differential equations) a perturbation $\delta_n^{(j)}$ of the uncoupled system

$$
\delta_{n+1}^{(j)} = f'(x_n^{(j)}, \mu_j) \delta_n^{(j)}, \qquad \delta_0 = 1.
\tag{21}
$$

Defining

$$
\delta x_n^{(j)} = z_n^{(j)} \delta_n^{(j)},
\tag{22}
$$

we obtain from Eqs. (20) and (21)

$$
z_{n+1}^{(j)} - z_n^{(j)} = k \frac{g(x_n^{(j)})}{\delta_{n+1}^{(j)}} \sum_{m=1}^{N} A_{jm} q'(x_n^{(m)}) \delta x_n^{(m)},
\tag{23}
$$

from which we can obtain

$$
z_{n+1}^{(j)} = k \sum_{p=0}^{n} \frac{g(x_p^{(j)})}{\delta_{p+1}^{(j)}} \sum_{m=1}^{N} A_{jm} q'(x_p^{(m)}) \delta x_p^{(m)} + z_0^{(j)}.
\tag{24}
$$

Multiplying both sides of this equation by $A_{lj} q'(x_{n+1}^{(j)}) \delta_{n+1}^{(j)}$, summing over j and using Eq. (22), we obtain

$$
\begin{aligned}
\Gamma_{n+1}^{(l)} &= k \sum_{j=1}^{N} A_{lj} \sum_{p=0}^{n} \frac{q'(x_{n+1}^{(j)}) \delta_{n+1}^{(j)} g(x_p^{(j)}) \Gamma_p^{(j)}}{\delta_{p+1}^{(j)}} \\
&\quad + z_0 \sum_{j=1}^{N} A_{lj} q'(x_{n+1}^{(j)}) \delta_{n+1}^{(j)},
\end{aligned}
\tag{25}
$$

where

$$
\Gamma_n^{(j)} = \sum_{m=1}^{N} A_{jm} q'(x_n^{(m)}) \delta x_n^{(j)}.
\tag{26}
$$

Assuming that perturbations from the incoherent state grow exponentially, $\Gamma_n^{(j)} = \gamma^{(j)} \eta^n$, and using $z_0 = \delta x_0$, we obtain

$$
\begin{aligned}
\gamma^{(l)} &= k \sum_{j=1}^{N} A_{lj} \gamma^{(j)} \sum_{p=0}^{n} \frac{q'(x_{n+1}^{(j)}) \delta_{n+1}^{(j)} g(x_p^{(j)}) \eta^{p-n-1}}{\delta_{p+1}^{(j)}} \\
&\quad + \delta x_0 \sum_{j=1}^{N} A_{lj} q'(x_{n+1}^{(j)}) \frac{\delta_{n+1}^{(j)}}{\eta^{n+1}}.
\end{aligned}
\tag{27}
$$

Since $\delta_n^{(j)}$ is a perturbation from the uncoupled system, it grows exponentially for large n as $\chi(\mu_j, x_0)^n$, where $\chi(\mu, x_0)$ is the Lyapunov exponent for the uncoupled system with parameter vector μ and initial condition x_0. If

$$
|\eta| > \chi \equiv \sup_{x_0, \mu} \chi(x_0, \mu),
\tag{28}
$$

then for large n we can neglect the last term in Eq. (27) and obtain

$$\gamma^{(l)} = k \sum_{j=1}^{N} A_{lj} \gamma^{(j)} \sum_{p=0}^{n} \frac{q'(x_{n+1}^{(j)}) \delta_{n+1}^{(j)} g(x_p^{(j)}) \eta^{p-n-1}}{\delta_{p+1}^{(j)}}. \quad (29)$$

In order to proceed further, we will use again the assumptions of large numbers of neighbors per node and statistical independence of the network and the vector of parameters. As we did in Eq. (19), we approximate Eq. (29) by

$$\gamma^{(l)} = k \left\langle \sum_{p=0}^{n} \frac{q'(x_{n+1}^{(j)}) \delta_{n+1}^{(j)} g(x_p^{(j)}) \eta^{p-n-1}}{\delta_{p+1}^{(j)}} \right\rangle \sum_{j=1}^{N} A_{lj} \gamma^{(j)}. \quad (30)$$

[If the sum over j in Eq. (29) is imagined as approximating N times the expected value of a product of two random variables, Eq. (30) approximates N times the product of the two expected values as suggested by our assumption of their independence.]

The average in Eq. (30) is the quantity $Z(\eta)$ introduced in Section 2.1. If we define $\gamma = [\gamma^{(1)}, \gamma^{(2)}, \ldots, \gamma^{(N)}]^T$, Eq. (30) can be rewritten as

$$\gamma = k Z(\eta) A \gamma. \quad (31)$$

Thus, γ is an eigenvector u_l of A (u_l is the lth eigenvector of A), and we have

$$1 = k \lambda_l Z(\eta), \quad (32)$$

where λ_l is its corresponding eigenvalue. The onset of instability of the incoherent state corresponds to $|\eta| = 1$, or $\eta = e^{i\omega}$. Thus, network mode l becomes unstable at a critical coupling strength satisfying

$$k_c^{(l)} = (\lambda_l Z(e^{i\omega_c}))^{-1}, \quad (33)$$

where the critical frequency at instability onset ω_c is found by $\mathrm{Im}(\lambda_l Z(e^{i\omega})) = 0$, where Im denotes the imaginary part. We are interested in the solutions k_c of Eq. (33) with the smallest magnitude. If there is more than one value of ω that yields $\mathrm{Im}(\lambda_l Z(e^{i\omega})) = 0$, the value ω_c that yields k_c with the smallest positive (negative) value is the one with the largest positive (negative) value of $\mathrm{Re}(\lambda_l Z(e^{i\omega}))$. Typically, the critical coupling strengths with the smallest magnitude correspond to the mode associated to the eigenvalue of largest magnitude, which is usually real (e.g., for symmetric or nonnegative matrices [20]). In this case the critical frequency ω_c is found from $\mathrm{Im}[Q(e^{i\omega})] = 0$ and is independent of the network. Thus, the positive critical coupling strength is given by $k_c = (Z(e^{i\omega_c})\lambda_+)^{-1}$, where $Z(e^{i\omega_c})$ depends exclusively on the dynamics of the uncoupled oscillators and their parameter distribution function $p(\mu)$, while the largest positive eigenvalue λ_+ depends exclusively on the network (and similarly for the negative critical coupling strength).

We will now comment on how the incoherent state loses stability as the magnitude of the coupling strength is increased from zero. Before considering the general case, we will discuss the globally coupled case, in which $A_{jm} = 1/N$ for all j, m. In this case, there is one positive eigenvalue, $\lambda_N = 1$, while the rest of the eigenvalues are 0. By Eq. (33), these modes do not become unstable at finite coupling strengths. Therefore, in this case the onset of instability is determined by λ_N only, and

typically the incoherent state is stable in an interval (k_-, k_+) where k_+ and k_- are the positive ($k_+ > 0$) and negative ($k_- < 0$) solutions of $1 = k Z(e^{i\omega})$ with smallest magnitude. Note that $k_\pm$ can be 0 or $\pm\infty$. Now we return to the general coupling case, still employing the notation $k_\pm$ to indicate the critical coupling strengths in the all-to-all case. In the case of general coupling, the largest positive eigenvalue λ_+ and the most negative eigenvalue λ_- might be of comparable size, especially if A has negative entries (since the trace of A is zero, there must be at least one negative and one positive eigenvalue). As the coupling constant k is increased from zero, the incoherent state can become unstable either because $k\lambda_+ = k_+$, or because $k\lambda_- = k_-$, whichever occurs first, and similarly as k is decreased from zero.

3.2. Networks of coupled continuous time oscillators described by ordinary differential equations

A similar analysis can be performed for the case of continuous time oscillators coupled in a network. In this section we will extend the results in Ref. [18], which deals with globally coupled continuous time oscillators, to the case of general connectivity. Since the analysis is analogous to that for maps in the previous section, we will just state the essential results.

The system under consideration is analogous to Eq. (18),

$$\dot{y}^{(j)}(t) = F(y^{(j)}(t), \mu_j) + W^{(j)}(t)$$
$$+ K(y^{(j)}(t)) \sum_{m=1}^{N} A_{jm}[Q(y^{(m)}(t)) - \langle Q(y) \rangle], \quad (34)$$

where, except for the fact that the continuous variable t has replaced the discrete variable n and we use upper case letters to avoid confusion, the conventions are similar to those of the previous section. As before, we assume that nodes have a large number of neighbors and the randomly chosen parameter vectors μ_j are not correlated with the network structure. Thus the incoherent state is an approximate solution of system (34).

In analogy to δ_n in Eq. (21), we define the fundamental matrix $M_j(t, y_j(0))$ for the linearized noiseless uncoupled version of Eq. (34),

$$\dot{M}_j(t, y_j(0)) = DF(y_j(t), \mu_j) M_j(t, y_j(0)). \quad (35)$$

If we consider a small perturbation $\delta y^{(j)}(t)$ from the incoherent state $y^{(j)}(t)$ and linearize Eq. (34), we obtain

$$\dot{\delta} y^{(j)}(t) = DF(y^{(j)}(t), \mu_j) \delta y^{(j)}(t)$$
$$+ K(y^{(j)}(t)) \sum_{m=1}^{N} A_{jm} DQ(y^{(m)}(t)) \delta y^{(m)}(t). \quad (36)$$

In terms of M, the solution is given by

$$\delta y^{(j)}(t) = \delta y^{(j)}(0)$$
$$+ \int_0^t M(t, y^{(j)}(0)) M^{-1}(\tau, y^{(j)}(0)) K(y^{(j)}(\tau))$$
$$\times \sum_{m=1}^{N} A_{jm} DQ(y^{(m)}(\tau)) \delta y^{(m)}(\tau) d\tau. \quad (37)$$

Using $M(t, y(0))M^{-1}(\tau, y(0)) = M(t - \tau, y(\tau))$ and defining $T \equiv t - \tau$, we get

$$
\begin{aligned}
\delta y^{(j)}(t) = {}& \delta y^{(j)}(0) \\
& + \int_0^t M(T, y^{(j)}(t - T)) K(y^{(j)}(t - T)) \\
& \times \sum_{m=1}^N A_{jm} DQ(y^{(m)}(t - T)) \delta y^{(m)}(t - T) \mathrm{d}T. \quad (38)
\end{aligned}
$$

In analogy with $\Gamma_n^{(j)}$ in Section 3.1, we define $\Psi^{(j)}(t) = \sum_m A_{jm} DQ(y^{(m)}(t)) \delta y^{(m)}(t)$ and assume exponential growth, $\Psi^{(j)}(t) = \psi^{(j)} \mathrm{e}^{st}$. Multiplying Eq. (39) by $A_{ij} DQ(y^{(j)}(t))$ and summing over j, we get

$$
\begin{aligned}
\psi^{(i)} = {}& \sum_{j=1}^N DQ(y^{(j)}(t)) \int_0^t M(T, y^{(j)}(t - T)) \\
& \times K(y^{(j)}(t - T)) \mathrm{e}^{-sT} \mathrm{d}T A_{ij} \psi^{(j)}, \quad (39)
\end{aligned}
$$

where we have neglected $\sum_j A_{ij} DQ(y^{(j)}(t)) \delta y^{(j)}(0) \mathrm{e}^{-st}$ in the right hand side after letting $t \to \infty$ (we assume DQ is bounded).

Assuming, as in Section 3.1, that the terms $A_{ij} \psi^{(j)}$ in the sum over j in Eq. (39) are statistically independent from the rest, we obtain a dispersion relation for the mode corresponding to eigenvalue λ_l of the connectivity matrix A,

$$
\det\{I - \lambda_l \hat{M}(s)\} = 0, \quad (40)
$$

where

$$
\begin{aligned}
\hat{M}(s) = \Bigg\langle {}& \int_0^\infty DQ(y(t)) K(y(t - T)) \\
& \times M(T, y(t - T)) \mathrm{e}^{-sT} \mathrm{d}T \Bigg\rangle. \quad (41)
\end{aligned}
$$

We note that $\hat{M}$ depends only on the dynamics of the individual, uncoupled oscillators, and the effect of the network is to scale the all-to-all critical coupling strength for mode j by a factor $1/\lambda_l$.

The same caveats that were noted for the maps apply here; for $\mathrm{Re}(s) > \chi > 0$ the integrand decays fast enough with time that the average and the integral can be interchanged. If $\langle DQ(y(t)) K(y(t - T)) M(T, y(t - T)) \rangle$ decays exponentially, the time integral converges for some $\epsilon < \mathrm{Re}(s) < 0$ and the result can then be analytically continued to $\mathrm{Re}(s) \geq 0$ (necessary to investigate the onset of instability of the incoherent state).

We now comment on why is it expected that $\langle DQ(y(t)) K(y(t - T)) M(T, y(t - T)) \rangle$ decays with time. If we imagine that the same small displacement δy_0 is applied to each point initially distributed according to the natural measure of the attractors, and then the average position of the evolved perturbations is compared with the centroid of the attractor, the difference is given by $\langle M(T, y) \rangle \delta y_0$. Although each perturbation might grow, the perturbed points will rapidly redistribute themselves on the attractor so that their average position coincides with that of the natural measure of the attractor, and thus we conclude that $\langle M(T, y) \rangle \to 0$. The presence of the smooth, bounded functions K and DQ is not expected to change this general picture. For a more detailed discussion of this issue, and some numerical examples, we refer the reader to [18]. We note that Eq. (40) is the same as the result obtained in [18] if λ_l is replaced by $\lambda_l = 1$ (the largest eigenvalue of an equally coupled all-to-all network, $A_{jm} = 1/N$), and DQ and K are set to I and $-kI$, respectively, where I is the identity matrix.

4. Examples

In this section we will consider various examples of the transition to coherence in networks of coupled dynamical systems. We will first present examples for maps and then an example for time continuous oscillators.

In order to quantify the coherence, we define the order parameter r by

$$
r^2 = \left\langle \left(\frac{\sum_{m=1}^N \overline{d}_m [q(x_n^{(m)}) - \langle q(x) \rangle]}{\sum_{m=1}^N \overline{d}_m} \right)^2 \right\rangle_n, \quad (42)
$$

where $\langle \cdot \rangle_n$ denotes a time average and the out-degree is defined by $\overline{d}_m = \sum_{j=1}^N A_{jm}$. Note that the numerator can be written as

$$
\sum_{j=1}^N \sum_{m=1}^N A_{jm} [q(x_n^{(m)}) - \langle q(x) \rangle], \quad (43)
$$

and, therefore, r measures the *rms* of the coupling term in Eq. (18), except for the factor $g(x^{(j)})$. Thus, the incoherent state corresponds to $r \approx 0$. We will investigate what happens to r as the coupling strength k is increased past the critical values predicted by the theory.

In our numerical examples, the order parameter r will be computed using Eq. (42) with the values of $x_n^{(j)}$ obtained from the solution of Eq. (18). The time average will be computed using 1000 iterations after the initial transients have disappeared (close to the transition these transients may last for a very long time, and it may therefore be difficult to estimate the asymptotic value of r).

As an example, we consider for the functions in Eq. (18),

$$
f(x_n^{(j)}, \mu_j) = 2x_n^{(j)} + \mu_j, \quad (44)
$$
$$
q(x) = \cos x, \quad (45)
$$

and

$$
g(x) = \sin(2x) + \sin(4x), \quad (46)
$$

where we note that this example was considered for the globally coupled case in Ref. [17]. Throughout we consider x as a variable on the unit circle, so mod 2π should be taken where appropriate. For the noise, we will consider Gaussian noise with zero mean and standard deviation σ.

In Ref. [17] $Z(\mathrm{e}^{\mathrm{i}\omega})$ was calculated for this case to be

$$
Z(\mathrm{e}^{\mathrm{i}\omega}) = -\frac{1}{2} \left(\frac{\mathrm{e}^{-\sigma^2/2}}{\mathrm{e}^{\mathrm{i}\omega}} \langle \cos(\mu) \rangle + 2 \frac{\mathrm{e}^{-5\sigma^2/2}}{\mathrm{e}^{2\mathrm{i}\omega}} \langle \cos(3\mu) \rangle \right), \quad (47)
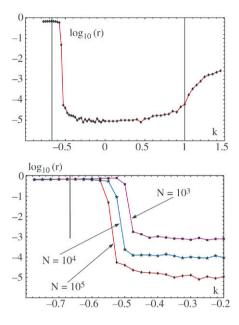$$

Fig. 1. Logarithm of the order parameter, $\log_{10} r$, as a function of the coupling strength k for example 1 (identical noiseless maps), for a scale-free network with degree distribution $P(d) \propto d^{-3}$ if $d \geq 100$, and 0 otherwise, for (a) $N = 10^5$, and (b) $N = 10^5$, $N = 10^4$, $N = 10^3$. The vertical lines indicate the theoretical values for the critical coupling strength.

from which we will obtain $k_c \lambda_l$ by solving Eq. (33). We remark that we are making use of the separation of the problem into a network part and a dynamics part, and using existing results for the dynamics part obtained in Ref. [17] for the globally coupled case.

We will numerically consider the following examples. Below we list each example with the corresponding theoretical results for the critical coupling strength.

(1) Identical noiseless maps with $\sigma = 0$, and $p(\mu) = \delta(\mu)$. The solution of Eq. (33) yields for this example $k_c \lambda_l = 1$ and $k_c \lambda_l = -2/3$.
(2) Heterogeneous noisy maps with $\sigma = 0.2$, $p(\mu) = 2$ if $0 \leq \mu < 1/2$ and 0 otherwise. In this example we obtain $k_c \lambda_l \approx 1.66$ and $k_c \lambda_l \approx -0.93$.
(3) Heterogeneous noisy maps with $\sigma = 0.1$, $p(\mu) \propto e^{-2\mu}$ for $\mu \geq 0$. In this example we obtain $k_c \lambda_l \approx 3.31$ and $k_c \lambda_l \approx -1.42$.

(Examples with identical noisy maps and heterogeneous noiseless maps can be found in Ref. [19].)

For the network connectivity, we consider scale-free networks, i.e., networks in which the out- and in-degree distributions satisfy $P(d) \propto d^{-\gamma}$ for $d \geq 100$ and 0 otherwise. We impose a lower cutoff $d \geq 100$ so that our assumption of a large number of neighbors per node is satisfied. In order to construct such networks we use the Random Graph model of Chung et al. [22]. In this method a desired degree sequence is generated first (in our case a sequence of integers d_j distributed according to $P(d) \propto d^{-\gamma}$ if $d \geq 100$, and 0 otherwise), and then the adjacency matrix elements A_{jm} are randomly chosen to be 1 with probability proportional to $d_j d_m$ and 0 otherwise. The diagonal elements of A are taken to be 0. The expected values of the in- and out-degrees are given by the prescribed sequence d_j.

This method has the property that it does not typically result in degree–degree correlations. We note, however, that the largest eigenvalue of A captures these correlations, and, therefore, we expect our theory to work in the presence of degree–degree correlations.

For scale-free networks as described above, the largest positive eigenvalue λ_+ is significantly larger than the magnitude of the most negative eigenvalue λ_- (for the network in Fig. 1 (a), for example, $\lambda_+ \approx 343.8$ and $\lambda_- \approx -43.1$). Moreover, for our examples k_+ and k_- are of the same order of magnitude (see values of $k_c \lambda_l$ above). Therefore, according to the discussion at the end of Section 3.1, the mode desynchronizing first will be the one associated to the largest positive eigenvalue. Consequently, the value of λ_l used to determine the critical coupling strengths k_c for these examples will be λ_+.

In Fig. 1(a) we show the logarithm of the order parameter as a function of k for example 1 (identical noiseless maps) and $N = 10^5$. The vertical lines indicate the values of the positive and negative values of the critical coupling strength predicted by the theory. We observe that the order parameter grows at values of the coupling strength close to the values predicted by the theory.

We note that the magnitude of k at which the transition takes place on the negative side is somewhat *smaller* than the prediction of the theory. We speculate that this is due to finite size effects: the form of the transition suggests that, as k is decreased past the negative critical value, the incoherent state ($r \approx 0$) suddenly loses its linear stability without a nearby attractor to capture the dynamics (as occurs in a subcritical pitchfork bifurcation). For finite values of N, the incoherent state is subject to fluctuations which, depending on the details of its basin of attraction, can push the dynamics to the basin of attraction of another attractor before the incoherent state loses its local stability. In order to test this hypothesis, we show in Fig. 1(b) a close up of the transition on the negative side for smaller values of N. As N is decreased, the transition occurs for smaller values of the magnitude of the coupling strength, thus suggesting that for $N \to \infty$ this effect should disappear. In fact, this issue is also present to a smaller extent in the globally coupled case. In that case it is numerically feasible to simulate much larger networks, and the discrepancy between theory and simulations is indeed extremely small for N of the order of 10^7 (see Fig. 3(a) of Ref. [17]).

In Fig. 2 we show the order parameter as a function of k for example 2 (heterogeneous noisy maps) with $N = 50\,000$ (the inset shows the same quantity on a logarithmic scale). Again, we observe that the order parameter grows for values of the coupling strength close to the values predicted by the theory. We also observe that the transitions on the positive and negative side seem qualitatively different, as in example 1. We find that in some cases (e.g., Fig. 2, negative side) the transition is quite sharp, suggesting that the incoherent state suddenly loses its stability without a nearby stable attractor to capture the dynamics, like occurs in subcritical pitchfork bifurcations. In other cases (e.g., Fig. 2, positive side), the transitions appear smoother, like occurs in transcritical and supercritical pitchfork or Hopf bifurcations. For the Lorentz oscillator example we
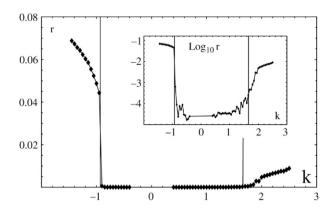
Fig. 2. Order parameter as a function of the coupling strength for example 2 (heterogeneous noisy maps), for a scale-free network with in- and out-degree distribution $P(d) \propto d^{-2.5}$ if $d \geq 100$, and 0 otherwise with $N = 10\,000$ (the inset shows the same quantity on a logarithmic scale). The vertical lines indicate the theoretical values for the critical coupling strength.
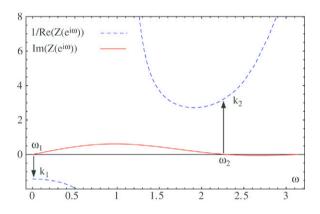

Fig. 3. Imaginary and inverse of the real parts of $Z(e^{i\omega})$ as a function of ω for example 3. The smallest positive (negative) coupling strength corresponds to $1/\text{Re}(Z(e^{i\omega}))$ for the value of ω for which $\text{Im}(Z(e^{i\omega})) = 0$ and $1/\text{Re}(Z(e^{i\omega}))$ is less positive (negative). In this example, the location of ω_2 and the value of k_2 are very sensitive to perturbations to Z.
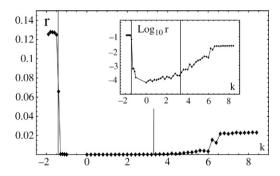

Fig. 4. Order parameter as a function of the coupling strength for example 3 (heterogeneous noisy maps), for a scale-free network with in- and out-degree distribution $P(d) \propto d^{-2.5}$ if $d \geq 100$, and 0 otherwise with $N = 10\,000$ (the inset shows the same quantity on a logarithmic scale). The vertical lines indicate the theoretical values for the critical coupling strength.

will discuss below, we will be able to rule out the transcritical bifurcation due to the symmetry of the dynamical system [18]. However, we will in general not attempt to characterize the precise nature of the transitions.

In the previous two examples, the agreement between the theory and the numerical simulations was good. However, there are some cases in which the approximations we have made (e.g., in assuming that each node has a large number of neighbors) can more significantly affect the result. We include example 3 in order to illustrate these potential problems. In Fig. 3 we show $1/\text{Re}(Z(e^{i\omega}))$ and $\text{Im}(Z(e^{i\omega}))$ as a function of ω. From the solution of Eq. (33), the smallest positive (negative) coupling strength (times λ_l) corresponds to $1/\text{Re}(Z(e^{i\omega}))$ for the value of ω for which $\text{Im}(Z(e^{i\omega})) = 0$ and $1/\text{Re}(Z(e^{i\omega}))$ is of smallest positive (negative) value. In this figure we observe that the smallest positive critical coupling strength, $k_2 \approx 3.31$, corresponding to the root ω_2 (see Fig. 3) of $\text{Im}(Z(e^{i\omega}))$, is very sensitive to the location of the root and could, upon a small perturbation to Z, increase to a value lying somewhere in $\sim (3, 8)$. On the other hand, the smallest negative coupling strength, $k_1 = -1.42$, is more robust against these effects. In Fig. 4 we show the order parameter as a function of k for this example (the inset shows the same quantity on a logarithmic scale). On the negative side, the order parameter grows for values of the coupling strength close to the values predicted by the theory, while on the positive side, the transition is not as well defined and apparently takes place at a coupling strength larger than the theoretical coupling strength $k_2 \approx 3.31$.

Besides cases like the previous example, there are other potentially problematic situations. In the globally coupled case studied in Ref. [17], it was found that the computation of $Z(\eta)$ fails to converge for an ensemble of identical noiseless logistic maps. This was tentatively attributed to the structural instability of the map and singularities in its invariant density, which make a perturbation approach questionable. Since the definition and numerical determination of $Z(\eta)$ in our case and for the globally coupled case are identical, the lack of convergence observed for this example in the globally coupled case will also occur in the case of a network. However, we note that a small amount of either noise or parameter heterogeneity was shown in [17] to restore the validity of the results. More generally, the numerical computation of Z and $\hat{M}$ is for most cases a nontrivial problem. Since these issues are intrinsic to the dynamics part of the problem and our interest is in the effect of the network structure on the dynamics, we will not study this in more detail. A more extensive discussion of these issues, and various methods for the numerical calculation of Z and $\hat{M}$, can be found in Refs. [17,18]. Another caveat is the crucial assumption in our analysis, allowing us to separate the dynamics and the network terms in Eq. (30), that the number of neighbors per node is large. For networks of sine coupled phase oscillators, we have noted in Ref. [7] that, if this assumption is not satisfied, the transition occurs at values of the coupling strength larger than those predicted by the theory.

As a final example we consider an ensemble of chaotic Lorenz oscillators. The dynamics of an individual uncoupled system is given by

$$\dot{y}_1 = \sigma(y_2 - y_1), \qquad \dot{y}_2 = \rho y_1 - y_2 - y_1 y_3,$$
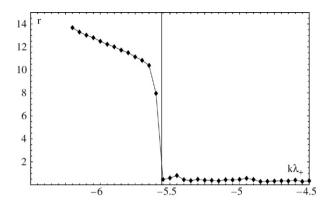$$\dot{y}_3 = -b y_3 + y_1 y_2, \tag{48}$$

Fig. 5. Order parameter as a function of the coupling strength for the heterogeneous ensemble of noiseless chaotic Lorenz oscillators, for a scale-free network with in- and out-degree distribution $P(d) \propto d^{-2.5}$ if $d \geq 100$, and 0 otherwise with $N = 5000$. The vertical line indicates the theoretical value for the critical coupling strength.

where, in the notation of Section 3.2, $y = (y_1, y_2, y_3)^T$, and the parameter vector is given by $\mu = (\sigma, b, \rho)^T$. As in Ref. [18], we will consider an ensemble of noiseless Lorenz oscillators with parameters given by $\sigma = 10, b = 8/3$, and ρ uniformly distributed in $(28, 52)$. This ensemble mainly has members with chaotic dynamics. We will use the coupling matrix K defined by $K_{11} = 1$ and $K_{ij} = 0$ for $(i, j) \neq (1, 1)$. Also, we will consider $Q(y) = y$. As before, we make use of the facts that the dynamics and network parts of the problem separate and that $\hat{M}(s)$ was already calculated for the globally coupled case. In Ref. [18], the critical coupling strength was found to be $k \approx -5.56$ (with no transition for $k > 0$), and thus for the case of a network we have $k\lambda_+ \approx -5.56$. We consider a scale-free network constructed as described above with $N = 5000$ and $\gamma = 2.5$.

For the continuous case, we define the order parameter as in Eq. (42),

$$r^2 = \left\langle \left\| \frac{\sum_{m=1}^{N} \overline{d}_m [Q(y^{(m)}) - \langle Q(y) \rangle]}{\sum_{m=1}^{N} \overline{d}_m} \right\|^2 \right\rangle_t , \tag{49}$$

where $\|z\| = (\sum_{m=1}^{N} z_n^2)^{1/2}$ and $\langle \cdot \rangle_t$ denotes a time average. In Fig. 5 we show the order parameter r (diamonds) as a function of $\lambda_+ k$. We see that at the critical coupling strength predicted by the theory (vertical line), the order parameter increases as predicted.

5. Discussion

We have found that, under very general conditions, collections of heterogeneous dynamical systems interacting in a complex network undergo a transition from incoherence to coherence as the coupling strength exceeds a critical value. The critical coupling strength is given by $k_c = (\mu\lambda)^{-1}$, where λ is the largest eigenvalue of the adjacency matrix of the network and μ depends on the dynamics of the individual, uncoupled dynamical systems. Therefore, the topology of the network and the dynamics of the oscillators affect the critical coupling strength independently. In this respect our result is in the same

spirit as that of the 'master stability function' of Pecora and Carroll [21]. The main difference is that we consider the onset of synchronism from an incoherent state of many potentially different chaotic or periodic systems, while Pecora and Carroll consider the stability of a fully synchronized state of identical oscillators.

Our results show that the largest eigenvalue of the adjacency matrix of the network plays a key role in determining the transition to coherence. There are other situations where it also plays an important role in network dynamical processes, for example in epidemic onset [23], percolation in networks [24], stability [25] and others [26,20]. Thus, an understanding of network properties determining the largest eigenvalue of the adjacency matrix could be used to control dynamics on networks.

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