APPM 4/5560

"Periodicity is a Class Property"

Periodicity is a class property. ie: If $i \leftrightarrow j$, then i and j have the same period.

Proof:

- To show this this, I will use the notation

$$a|b,$$

 to say that a divides evenly into b. We read $a|b$ as "a divides b". Equivalently, this means that that b is a multiple of a. (i.e. There exists a positive integer k such that $b = ka$.)

 For example, 3 divides 9 and 3 divides 21 but 3 does not divide 5. We write $3|9$, $3|21$, and $3 \nmid 5$.

 Note that if $a|b$, then b must be greater than or equal to a. If $a|b$ and $b|a$ then the integer k must be 1 and we can conclude that $a = b$.

 Now let's get to the proof...

- Let d_i and d_j be the periods of states i and j, respectively.

 Since $i \leftrightarrow j$, there exist integers m and n so that

$$p_{ij}^{(n)} > 0 \qquad \text{and} \qquad p_{ji}^{(m)} > 0.$$

- Since

$$p_{jj}^{(m+n)} \geq p_{ji}^{(m)} p_{ij}^{(n)} > 0,$$

 we must have that $d_j|(m+n)$ since, by definition of period, d_j is the greatest common divisor of all integers k such that $p_{jj}^{(k)} > 0$.

- Let u be any integer such that $p_{ii}^{u} > 0$. (We know such an integer exists since one example is given by $u = n + m$ since

$$p_{ii}^{(n+m)} \geq p_{ij}^{(n)} p_{ji}^{(m)} > 0.)$$

- Since

$$p_{jj}^{(m+u+n)} \geq p_{ji}^{(m)} p_{ii}^{(u)} p_{ij}^{(n)} > 0,$$

 we know that $d_j|(m+k+n)$.

- Now $d_j|(m+n)$ and $d_j|(m+u+n)$ implies that $d_j|u$. (Check this!)

- Since u was arbitrary, we now know that d_j divides ever power k such that $P_{ii}(k) > 0$. So, we can conclude that
$$d_j | d_i$$
since d_i is the greatest common divisor of all powers k such that $p_{ii}^{(k)} > 0$.
- A symmetric argument shows that
$$d_i | d_j.$$
- If both $d_i | d_j$ and $d_j | d_i$, we must have that $d_i = d_j$.
- Yay.