Periodicity is a class property. ie: If \( i \leftrightarrow j \), then \( i \) and \( j \) have the same period.

Proof:

- To show this, I will use the notation
  \[ a | b, \]
to say that \( a \) divides evenly into \( b \). We read \( a | b \) as “\( a \) divides \( b \)”. Equivalently, this means that \( b \) is a multiple of \( a \). (i.e. There exists a positive integer \( k \) such that \( b = ka \).)

For example, \( 3 \) divides \( 9 \) and \( 3 \) divides \( 21 \) but \( 3 \) does not divide \( 5 \). We write \( 3 | 9 \), \( 3 | 21 \), and \( 3 \nmid 5 \).

Note that if \( a | b \), then \( b \) must be greater than or equal to \( a \). If \( a | b \) and \( b | a \) then the integer \( k \) must be 1 and we can conclude that \( a = b \).

Now let’s get to the proof...

- Let \( d_i \) and \( d_j \) be the periods of states \( i \) and \( j \), respectively.

  Since \( i \leftrightarrow j \), there exist integers \( m \) and \( n \) so that
  \[ p_{ij}^{(n)} > 0 \quad \text{and} \quad p_{ji}^{(m)} > 0. \]

- Since
  \[ p_{jj}^{(m+n)} \geq p_{ji}^{(m)} p_{ij}^{(n)} > 0, \]
  we must have that \( d_j | (m+n) \) since, by definition of period, \( d_j \) is the greatest common divisor of all integers \( k \) such that \( p_{jj}^{(k)} > 0 \).

- Let \( u \) be any integer such that \( p_{ii}^{u} > 0 \). (We know such an integer exists since one example is given by \( u = n + m \) since
  \[ p_{ii}^{(n+m)} \geq p_{ij}^{(n)} p_{ji}^{(m)} > 0. \]

- Since
  \[ p_{jj}^{(m+u+n)} \geq p_{ji}^{(m)} p_{ii}^{(u)} p_{ij}^{(n)} > 0, \]
  we know that \( d_j | (m + k + n) \).

- Now \( d_j | (m + n) \) and \( d_j | (m + u + n) \) implies that \( d_j | u \). (Check this!)
• Since $u$ was arbitrary, we now know that $d_j$ divides every power $k$ such that $P_{ii}(k) > 0$. So, we can conclude that $d_j | d_i$ since $d_i$ is the greatest common divisor of all powers $k$ such that $p_{ii}^{(k)} > 0$.

• A symmetric argument shows that $d_i | d_j$.

• If both $d_i | d_j$ and $d_j | d_i$, we must have that $d_i = d_j$.

• Yay.