CONVOLUTIONS

Prop. 7.4 $f, g \in L^2(\mathbb{T})$, then 1) $\|f*g\|_\infty \le \|f\|_{L^2} \cdot \|g\|_{L^2}$

2) $(f*g) \in C(\mathbb{T})$

$$(f*g)(x) = \int f(x-y)g(y)\,dy$$

__Proof__: 1) $$|(f*g)(x)| \le \left| \int_{\mathbb{T}} f(x-y)\cdot g(y)\,dy \right|$$

$$\le \int_{\mathbb{T}} |f(x-y)| \cdot |g(y)|\,dy$$

$\tilde{f}(y) = f(x-y)$

$$= \int_{\mathbb{T}} |\tilde{f}(y)| \cdot |g(y)|\,dy$$

$$\le \|\tilde{f}\|_{L^2} \cdot \|g\|_{L^2}$$

$$= \|f\|_{L^2} \cdot \|g\|_{L^2}$$

$$= \|f\|_2 \cdot \|g\|_2.$$

2) Show $f*g$ is continuous.

$f \in L^2(\mathbb{T}) := \overline{C(\mathbb{T})}$, $\exists$ seq. $(f_k) \subseteq C$ s.t. $f_k \to f$, $g_k \to g$.

__Lemma__: $\forall k$, $f_k * g_k$ is continuous.

Pick $x, y \in \mathbb{T}$, $|f_k*g_k(x) - f_k*g_k(y)| = \left| \int_{\mathbb{T}} f_k(x-z)\,g_k(z) - \int_{\mathbb{T}} f_k(y-z)g_k(z)dz \right|$

$$\le \int_{\mathbb{T}} |f_k(x-z) - f_k(y-z)| \cdot |g_k(z)|\,dz$$

deduce f_k is uniformly continuous.

$\exists \delta > 0$ st. $\forall x', y'$ st. $|x'-y'| < \delta$ $\Rightarrow |f_k(x') - f_k(y')| < \varepsilon$

$x' = x - z$
$y' = y - z$ $\Rightarrow < \varepsilon$

z bounded.

$$\le \int_{\mathbb{T}} \varepsilon \cdot |g_k(z)|\,dz \le 2\pi\varepsilon \cdot \|g_k\|_\infty \quad \square.$$

Final step: claim $f_k * g_k$ is Cauchy w.r.t. $\|\cdot\|_\infty$ norm.

Limit is $f*g$. Use $C(\mathbb{T})$ are complete $\|\cdot\|_\infty$.

$\Rightarrow f*g$ is continuous.

$$\|f_j*g_j - f_k*g_k\|_\infty \le \|(f_j - f_k)*g_j\|_\infty + \|f_k * (g_j - g_k)\|_\infty$$

$$\le \underbrace{\|f_j - f_k\|_2}_{\text{small}} \|g_j\|_2 + \|f_k\|_2 \underbrace{\|g_j - g_k\|_2}_{\text{small}}. \quad \square$$

$f_k \to f$ in L_2

bounded.

Uniformly bdd?

$g_j \to g$, $\|g\|_2 < \infty$