PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
August 2019

- You have three hours to complete this exam.
- Each problem is worth 25 points.
- Work only four of the five problems problems.
- You must mark which four that you choose—only four will be graded.
- Start each problem on a new page.

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1. **Method of characteristics.** Suppose that $u(x,t)$ is defined by a PDE and by initial values (at $t = 0$):

$$\begin{aligned} \partial_t u + t\partial_x u = u &\quad \text{for all } x, \text{ and } t > 0. \\ u(x,0) = -x &\quad \text{for all } x, \text{ with } t = 0. \end{aligned}$$

   (a) Sketch a few of the characteristic curves in the (t,x)-plane for $t > 0$, and label them.
   (b) Find $u(x,t)$as explicitly as possible in the region in which $u(x,t)$ is defined.
   (c) State whether the characteristics ever cross for $t > 0$. If they cross, find a time (t) and location (x) where they cross, and do **not** answer 1d.
   (d) If the characteristics **never** cross, then evaluate $u(x,t)$ at $\{x = 1, t = 1\}$.

2. **Fourier Series.**
   Let $f(x) = \sin\{\pi\,|x|\}$, $-1 \le x \le 1$.

   (a) Sketch $f(x)$ on $-1 \le x \le 1$.
   (b) Find the first four nonzero terms in the Fourier series for $f(x)$.
   (c) Does the Fourier series fail to converge to $f(x)$ anywhere in $[-1,1]$? If so, where? Justify your answer.

3. **Wave Equation.** Let $v(x,t)$ denote the solution of:

$$\begin{aligned} &v_{tt}(x,t) = c^2 v_{xx}(x,t) + 2\sin(x)\cos(ct), -\pi < x < \pi,\ t > 0,\ c > 0 \\ &v(-\pi,t) = v(\pi,t) = 0,\ t > 0, \\ &v(x,0) = \cos(\tfrac{x}{2}),\ v_t(x,0) = 0. \end{aligned} \tag{1}$$

(a) Find $v(x,t)$ for $t > 0$, $-\pi < x < \pi$.

(b) Is $v(x,t)$ periodic in time? (Yes or No)

(c) If Yes, find the period (in time) of the motion. If No, is there a time $t > 0$ when $v(x,t) = v(x,0)$ for all $-\pi < x < \pi$? If so, find the first such time after $t = 0$.

4. **Elliptic Problem.**

In the following, the 2-norm will be assumed, i.e., $|\bullet| := \|\bullet\|_2$.

(a) Construct the Green's function $G(\boldsymbol{x}, \boldsymbol{x}')$ for the Dirichlet problem:

$$\begin{aligned} \Delta G(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \quad &; \quad \boldsymbol{x} \in \mathbb{R}^2,\ |\boldsymbol{x}| < 1 \\ G(\boldsymbol{x}, \boldsymbol{x}') = 0 \quad &; \quad |\boldsymbol{x}| = 1 \end{aligned}$$

(b) Write down and justify the formula for smooth solutions of

$$\begin{aligned} \Delta u = 0 \quad &; \quad \boldsymbol{x} \in \mathbb{R}^2,\ |\boldsymbol{x}| < 1 \\ u(\boldsymbol{x}) = g(\boldsymbol{x}) \quad &; \quad |\boldsymbol{x}| = 1 \end{aligned}$$

where g is a smooth function on the unit circle.

(c) Use the maximum principle to prove the uniqueness of the solution in (b).

5. **Heat equation.**

(a) Consider $Q = \{(x,t) | 0 < x < L, t > 0\}$ and $\bar{Q}$ to be the closure of Q. Assume u and v are in $C(\bar{Q}) \cap C^2(Q)$ (), and are solutions to the heat equation ($\partial_t u = \partial_x^2 u$) on Q. Furthermore, suppose $u \leq v$ for $t = 0$, for $x = 0$, and for $x = L$. (Use the Maximum Principle) to show that $u \leq v$ on Q.

(b) More generally, consider functions u and v which solve $\partial_t u - \partial_x^2 u = f(x,t)$ and $\partial_t v - \partial_x^2 v = g(x,t)$ on Q. Furthermore, assume that $f \leq g$ on Q and $u \leq v$ for $t = 0$, for $x = 0$, as well as $x = L$. Show that $u \leq v$ on Q.

(c) Suppose v satisfies $\partial_t v - \partial_x^2 v \geq \sin(x)$ on $R = \{(x,t) | 0 < x < \pi, t > 0\}$. Moreover, assume $v(0,t) \geq 0$ and $v(\pi,t) \geq 0$ for all $t > 0$ and $v(x,0) \geq \sin(x)$ for all $0 \leq x \leq 1$. Then show that $v(x,t) \geq (1 - \boldsymbol{e}^{-t}) \sin(x)$ on R.