PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
August 2019

- You have three hours to complete this exam.
- Each problem is worth 25 points.
- Work only four of the five problems problems.
- You must mark which four that you choose—only four will be graded.
- Start each problem on a new page.

1. **Method of characteristics.** Suppose that \( u(x, t) \) is defined by a PDE and by initial values (at \( t = 0 \)):

\[
\begin{align*}
\partial_t u + t \partial_x u &= u & \text{for all } x, \text{ and } t > 0. \\
 u(x, 0) &= -x & \text{for all } x, \text{ with } t = 0.
\end{align*}
\]

(a) Sketch a few of the characteristic curves in the \((t, x)\)-plane for \( t > 0 \), and label them.
(b) Find \( u(x, t) \) as explicitly as possible in the region in which \( u(x, t) \) is defined.
(c) State whether the characteristics ever cross for \( t > 0 \). If they cross, find a time \((t)\) and location \((x)\) where they cross, and do not answer 1d.
(d) If the characteristics never cross, then evaluate \( u(x, t) \) at \( \{x = 1, t = 1\} \).

2. **Fourier Series.**

Let \( f(x) = \sin \{\pi |x|\}, -1 \leq x \leq 1. \)

(a) Sketch \( f(x) \) on \(-1 \leq x \leq 1. \)
(b) Find the first four nonzero terms in the Fourier series for \( f(x) \).
(c) Does the Fourier series fail to converge to \( f(x) \) anywhere in \([-1, 1]\)? If so, where? Justify your answer.

3. **Wave Equation.** Let \( v(x, t) \) denote the solution of:

\[
\begin{align*}
v_{tt}(x, t) &= c^2 v_{xx}(x, t) + 2 \sin(x) \cos(\pi t), -\pi < x < \pi, \ t > 0, \ c > 0 \\
v(-\pi, t) &= v(\pi, t) = 0, \ t > 0, \\
v(x, 0) &= \cos(\frac{x}{2}), \ v_t(x, 0) = 0.
\end{align*}
\]
(a) Find \( v(x, t) \) for \( t > 0, -\pi < x < \pi \).

(b) Is \( v(x, t) \) periodic in time? (Yes or No)

(c) If Yes, find the period (in time) of the motion. If No, is there a time \( t > 0 \) when \( v(x, t) = v(x, 0) \) for all \( -\pi < x < \pi \)? If so, find the first such time after \( t = 0 \).

4. Elliptic Problem.

In the following, the 2-norm will be assumed, i.e., \( |\cdot| := \|\cdot\|_2 \).

(a) Construct the Green’s function \( G(x, x') \) for the Dirichlet problem:

\[
\Delta G(x, x') = \delta(x - x') \quad ; \quad x \in \mathbb{R}^2, |x| < 1
\]

\[
G(x, x') = 0 \quad ; \quad |x| = 1
\]

(b) Write down and justify the formula for smooth solutions of

\[
\Delta u = 0 \quad ; \quad x \in \mathbb{R}^2, |x| < 1
\]

\[
u(x) = g(x) \quad ; \quad |x| = 1
\]

where \( g \) is a smooth function on the unit circle.

(c) Use the maximum principle to prove the uniqueness of the solution in (b).

5. Heat equation.

(a) Consider \( Q = \{(x, t)|0 < x < L, t > 0\} \) and \( \overline{Q} \) to be the closure of \( Q \). Assume \( u \) and \( v \) are in \( C(\overline{Q}) \cap C^2(Q) \), and are solutions to the heat equation \( (\partial_t u = \partial_x^2 u) \) on \( Q \). Furthermore, suppose \( u \leq v \) for \( t = 0 \), for \( x = 0 \), and for \( x = L \). (Use the Maximum Principle) to show that \( u \leq v \) on \( Q \).

(b) More generally, consider functions \( u \) and \( v \) which solve \( \partial_t u - \partial_x^2 u = f(x, t) \) and \( \partial_t v - \partial_x^2 v = g(x, t) \) on \( Q \). Furthermore, assume that \( f \leq g \) on \( Q \) and \( u \leq v \) for \( t = 0 \), for \( x = 0 \), as well as \( x = L \). Show that \( u \leq v \) on \( Q \).

(c) Suppose \( v \) satisfies \( \partial_t v - \partial_x^2 v \geq \sin(x) \) on \( R = \{(x, t)|0 < x < \pi, t > 0\} \). Moreover, assume \( v(0, t) \geq 0 \) and \( v(\pi, t) \geq 0 \) for all \( t > 0 \) and \( v(x, 0) \geq \sin(x) \) for all \( 0 \leq x \leq 1 \). Then show that \( v(x, t) \geq (1 - e^{-t}) \sin(x) \) on \( R \).