PARTIAL DIFFERENTIAL EQUATIONS PRELIMINARY EXAMINATION
August 2019

- You have three hours to complete this exam.
- Each problem is worth 25 points.
- Work only four of the five problems problems.
- You must mark which four that you choose—only four will be graded.
- Start each problem on a new page.

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1. **Method of characteristics.** Suppose that $u(x,t)$ is defined by a PDE and by initial values (at $t = 0$):

$$\begin{aligned} \partial_t u + t\partial_x u &= u \quad \text{for all } x\text{, and } t > 0. \\ u(x,0) &= -x \quad \text{for all } x\text{, with } t = 0. \end{aligned}$$

   (a) Sketch a few of the characteristic curves in the (t,x)-plane for $t > 0$, and label them.

   (b) Find $u(x,t)$as explicitly as possible in the region in which $u(x,t)$ is defined.

   (c) State whether the characteristics ever cross for $t > 0$. If they cross, find a time (t) and location (x) where they cross, and do **not** answer 1d.

   (d) If the characteristics **never** cross, then evaluate $u(x,t)$ at $\{x = 1, t = 1\}$.

Solution:

   (a) The characteristic curves are defined by $\frac{dx}{dt} = t$, for $t > 0$, all real x.

$$\Longrightarrow x(t) = x(0) + \frac{t^2}{2}\,.$$

   (b) Along every characteristic curve, $\frac{dx}{dt} = t$ for $t > 0$, so the PDE becomes an ODE for

$$u(x(t),t) : \frac{du}{dt} = \partial_t u + \left(\frac{dx}{dt}\right)\partial_x u = u\,.$$
$$\Longrightarrow u(x(t),t) = C(x(0))e^t = C(1/2)e^t\,.$$

(c) From 1a, every characteristic is half of a parabola, and the curves are identical, except for their starting values of $x = x(0)$ at $t = 0 \implies$ *the characteristics never cross each other.*

(d) From 1a, if follows that the characteristic that goes through $(x = 1, t = 1)$ goes through $x = x(0) = \frac{1}{2}$ at $t = 0$. $\implies u(x,t) = C(1/2)e = -1/2e$.

2. **Fourier Series.**

Let $f(x) = \sin\{\pi |x|\}$, $-1 \le x \le 1$.

(a) Sketch $f(x)$ on $-1 \le x \le 1$.

(b) Find the first four nonzero terms in the Fourier series for $f(x)$.

(c) Does the Fourier series fail to converge to $f(x)$ anywhere in $[-1, 1]$? If so, where? Justify your answer.

Solution:

(a) *Sketch*

(b)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) .$$

$$a_0 = \frac{2}{\pi}; \text{ for all } n \ge 1 ,$$

$$a_{2n-1} = 0,\ a_{2n} = \frac{2}{\left[1 - (2n)^2\right]\pi} .$$

(c) The series converges absolutely for $-1 \le x \le 1$, so it converges pointwise to $f(x)$ for $-1 \le x \le 1$.

3. **Wave Equation.** Let $v(x,t)$ denote the solution of:

$$\begin{aligned} &v_{tt}(x,t) = c^2 v_{xx}(x,t) + 2\sin(x)\cos(ct), -\pi < x < \pi,\ t > 0,\ c > 0 \\ &v(-\pi,t) = v(\pi,t) = 0,\ t > 0, \\ &v(x,0) = \cos(\tfrac{x}{2}),\ v_t(x,0) = 0. \end{aligned} \tag{1}$$

(a) Find $v(x,t)$ for $t > 0$, $-\pi < x < \pi$.

(b) Is $v(x,t)$ periodic in time? (Yes or No)

(c) If Yes, find the period (in time) of the motion. If No, is there a time $t > 0$ when $v(x,t) = v(x,0)$ for all $-\pi < x < \pi$? If so, find the first such time after $t = 0$.

Solution:

(a) Verify directly that $v(x,t) = \cos(\frac{x}{2})\cos(\frac{ct}{2}) + t\sin(x)\sin(ct)$ solves the problem in a. To derive this solution:

i. Solve the homogeneous problem: $w_{tt} = c^2 w_{xx}$, $-\pi < x < \pi, t > 0$, with $w(-\pi,t) = w(\pi,t) = 0, t > 0$.
Separate variables: $w = F(x)G(t), \Longrightarrow F(x)G''(t) = c^2 G(t)F''(x)$
$\Longrightarrow \dfrac{G''(t)}{c^2G(t)} = \dfrac{F''(x)}{F(x)} = K$ where $K = const$. The x-equation and boundary conditions are:
$F''(x) = KF(x), F(-\pi) = F(\pi) = 0$
First, show that there are no nonzero solutions for $K > 0$ or for $K = 0$ with given boundary condition.
For $K = -\lambda^2 < 0$, $F(x) = A\sin(\lambda(x+\pi)) + B\cos(\lambda(x+\pi))$,
$F(-\pi) = 0 \Longrightarrow B = 0$ and $F(\pi) = 0 \Longrightarrow \lambda = \lambda_n = \frac{n}{2}, n = 1, 2, 3, \cdots$, and $F_n = \sin(\lambda_n(x+\pi))$.
With $\lambda_n = \frac{n}{2}$ and we solve the G equation to get $G_n = a_n\sin(c\lambda_n t) + b_n\cos(c\lambda_n t)$.
$u_n(x) = \sin(\frac{n}{2}(x+\pi))(a_n\sin(\frac{cn}{2}t) + b_n\cos(\frac{cn}{2}t))$ is a solution.
The general (formal) solution of the homogeneous problem is
$\displaystyle\sum_{n=1}^{\infty}\sin(\frac{n}{2}(x+\pi))(a_n\sin(\frac{cn}{2}t) + b_n\cos(\frac{cn}{2}t))$.
Notice that $u_1(x,t) = \sin(\frac{x+\pi}{2})(a\sin(\frac{ct}{2}) + b\cos(\frac{ct}{2})) = \cos(\frac{x}{2})(a\sin(\frac{ct}{2}) + b\cos(\frac{ct}{2}))$ it is easy to see that the solution of our homogeneous problem is given by $b = 1$ and $a = 0$:

$$w(x,t) = \cos(\frac{x}{2})\cos(\frac{ct}{2}).$$

ii. Next, address the forced problem. The spatial structure of the forced problem is simple, and sin (x) satisfies the boundary conditions, so we can look for a particular solution of the forced problem of the form:

$$u(x,t) = \sin(x)Q(t)$$

Substituting this into the PDE: $\Longrightarrow Q''(t) = -c^2Q(t) + 2cos(ct)$. The general solution of this ODE is: $Q(t) = t\sin(ct) + a\sin(ct) + b\cos(ct)$ with the initial condition $Q(0) = Q'(0) = 0$ we get:

$$Q(t) = t\sin(ct).$$

Thus the complete solution of the problem is:

$$v(x,t) = w(x,t + u(x,t) = \cos(\frac{x}{2})\cos(\frac{ct}{2}) + t\sin(x)\sin(ct).$$

(b) $v(x,t)$ is NOT periodic, because of the second term.

(c) $v(x,t) = v(x,0) = \cos(\frac{x}{2})$ at $ct = 4\pi$.

4. **Elliptic Problem.**

In the following, the 2-norm will be assumed, i.e., $|\bullet| := \|\bullet\|_2$.

(a) Construct the Green's function $G(\boldsymbol{x},\boldsymbol{x}')$ for the Dirichlet problem:

$$\begin{aligned} \Delta G(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \quad &; \quad \boldsymbol{x} \in \mathbb{R}^2,\ |\boldsymbol{x}| < 1 \\ G(\boldsymbol{x},\boldsymbol{x}') = 0 \quad &; \quad |\boldsymbol{x}| = 1 \end{aligned}$$

(b) Write down and justify the formula for smooth solutions of

$$\begin{aligned} \Delta u = 0 \quad &; \quad \boldsymbol{x} \in \mathbb{R}^2,\ |\boldsymbol{x}| < 1 \\ u(\boldsymbol{x}) = g(\boldsymbol{x}) \quad &; \quad |\boldsymbol{x}| = 1 \end{aligned}$$

where g is a smooth function on the unit circle.

(c) Use the maximum principle to prove the uniqueness of the solution in (b).

Solution:

(a) A homogeneous solution of Laplace's equation in $\mathbb{R}^2$ is

$$\Phi(r) = \frac{1}{2\pi}\ln(r), \qquad r = |\boldsymbol{x}|.$$

Using the method of images, we utilize the image point $\boldsymbol{x}'^* = \boldsymbol{x}'/|\boldsymbol{x}'|^2$, so for any $\boldsymbol{x}$ on the unit circle, $|\boldsymbol{x}-\boldsymbol{x}'|^2 = |\boldsymbol{x}'|^2 |\boldsymbol{x}-\boldsymbol{x}'^*|$, so the Green's function for the unit disk is

$$G(\boldsymbol{x},\boldsymbol{x}') = \Phi(|\boldsymbol{x}-\boldsymbol{x}'|) - \Phi(|\boldsymbol{x}-\boldsymbol{x}'^*|) - \Phi(|\boldsymbol{x}'|) = \frac{1}{4\pi}\ln\frac{r^2 + r'^2 - 2rr'\cos(\theta-\theta')}{r^2r'^2 + 1 - 2rr'\cos(\theta-\theta')}$$

(b) Justified using the Green's representation theorem, we can identify the solution using our result from (a), and mapping the boundary condition to polar coordinates: $u(1,\theta) = g(\theta)$. We know then the unit normal is in the radial direction, so for $\boldsymbol{x}' = (1, \theta')$,

$$\frac{\partial G(\boldsymbol{x}, \boldsymbol{x}')}{\partial \mathbf{n}'} = \frac{\partial G(r, \theta, 1, \theta')}{\partial r'} = \frac{1}{2\pi} \frac{1 - r^2}{r^2 + 1 - 2r\cos(\theta - \theta')}$$

Applying the Green's representation formula, we know the solution is

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{r^2 + 1 - 2r\cos(\theta - \theta')} g(\theta') \boldsymbol{d}\theta'$$

(c) Let $B = \{(r,\theta) | 0 \leq r < 1, 0 \leq \theta < 2\pi\} \in \mathbb{R}^2$. If we assume for a set B and its closure $\bar{B}$ that $u \in C^2(B) \cap C(\bar{B})$ and $\Delta u = 0$ in B, then $|u| \leq \max_{\partial B} |u| = \max |g|$. If u_1 and u_2 are both solutions in $C^2(B) \cap C(\bar{B})$, then $w = u_1 - u_2$ solves the Dirichlet problem, so the maximum principle ensures $|w| \leq 0$, so $w \equiv 0$, so $u_1 \equiv u_2$.

5. **Heat equation.**

(a) Consider $Q = \{(x,t) | 0 < x < L, t > 0\}$ and $\bar{Q}$ to be the closure of Q. Assume u and v are in $C(\bar{Q}) \cap C^2(Q)$ (), and are solutions to the heat equation ($\partial_t u = \partial_x^2 u$) on Q. Furthermore, suppose $u \leq v$ for $t = 0$, for $x = 0$, and for $x = L$. (Use the Maximum Principle) to show that $u \leq v$ on Q.

(b) More generally, consider functions u and v which solve $\partial_t u - \partial_x^2 u = f(x,t)$ and $\partial_t v - \partial_x^2 v = g(x,t)$ on Q. Furthermore, assume that $f \leq g$ on Q and $u \leq v$ for $t = 0$, for $x = 0$, as well as $x = L$. Show that $u \leq v$ on Q.

(c) Suppose v satisfies $\partial_t v - \partial_x^2 v \geq \sin(x)$ on $R = \{(x,t) | 0 < x < \pi, t > 0\}$. Moreover, assume $v(0,t) \geq 0$ and $v(\pi,t) \geq 0$ for all $t > 0$ and $v(x,0) \geq \sin(x)$ for all $0 \leq x \leq 1$. Then show that $v(x,t) \geq (1 - \boldsymbol{e}^{-t}) \sin(x)$ on R.

Solution:

(a) Take $w := u - v$, then w solves the heat equation. Fix $T > 0$, then $w \leq 0$ on $t = 0$, $x = 0$, and $x = 1$. Thus, by the maximum principle, it follows that $w \leq 0$ on $[0,1] \times [0,T]$. Choose $T > 0$ arbitrarily large, so $w \leq 0$ on $[0,1] \times [0,\infty)$, so $u \leq v$ on $[0,1] \times [0,\infty)$.

(b) Fix $T > 0$ and consider $Q_T := [0,1] \times [0,T]$. Look at $w := u - v$ on Q_T, then $\partial_t w - \partial_x^2 w \leq 0$ on $[0,1] \times (0,T]$ and $w \leq 0$ on $t = 0$, $x = 1$, and $x = 1$. We want to show $w \leq 0$ on Q_T. Define $w^\epsilon := w + \epsilon x^2$ and suppose there exists $(x_0, t_0) \in Q_T$ such that w^ϵ obtains its max, which does not lie on $t = 0$, $x = 0$, or $x = 1$:
1. If $(x_0, t_0) \in (0, L) \times (0, T)$ then $\partial_t w^\epsilon(x_0, t_0) = 0$ and $\partial_x^2 w(x_0, t_0) \leq 0$, so $\partial_t w^\epsilon(x_0, t_0) - \partial_x^2 w(x_0, t_0) \geq 0$. However, $\partial_t w^\epsilon - \partial_x^2 w^\epsilon = \partial_t w - \partial_x^2 w - 2\epsilon \leq -2\epsilon < 0$, contradiction.
2. If (x_0, t_0) lies on $t = T$, then $\partial_t w^\epsilon(x_0, t_0) \geq 0$ and $\partial_x^2 w^\epsilon(x_0, t_0) \leq 0$, so $\partial_t w^\epsilon(x_0, t_0) - \partial_x^2 w(x_0, t_0) \geq 0$. Again, this is a contradiction.
Therefore, the maximum of w^ϵ on Q_T can only be attained on $t = 0$, $x = 0$, or $x = 1$. Thus, $w \leq 0$ on Q_T. Taking $T \to \infty$, $w \leq 0$ on Q so $u \leq v$ on Q.

(c) Take $u = (1 - \boldsymbol{e}^{-t}) \sin x$, then $\partial_t u = \boldsymbol{e}^{-t} \sin x$ and $\partial_x^2 u = -(1 - \boldsymbol{e}^{-t}) \sin x$. Hence, $\partial_t u - \partial_x^2 u = \sin x$. Furthermore $u(0,t) = u(\pi, t) = 0$ for all $t > 0$. Furthermore, $u(x, 0) = 0 \leq \sin(x) = v(x, 0)$. From (b), note it follows $u \leq v$ on R, so $v \geq (1 - \boldsymbol{e}^{-t}) \sin x$ on R.