Preliminary Exam
Partial Differential Equations
1:00 - 4:00 PM, Wednesday, Jan. 5, 2022
Remotely

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.**

   (a) (15 points) Obtain an implicit solution to the following initial value problem (IVP):

$$u_t + u^2 u_x = 0, \qquad x \in (-\infty, \infty), \quad t > 0,$$
$$u(x,0) = g(x) = 1 - \mathrm{e}^{-|x|}.$$

Remember to check existence and uniqueness near the initial condition. Also, sketch the characteristics of the associated nonlinear wave solution in the (x,t) plane.

Solution: Begin by parameterizing the variables, generating set of ODEs

$$\frac{dx}{d\tau} = z^2, \qquad x_0(s) = s,$$
$$\frac{dt}{d\tau} = 1, \qquad t_0(s) = 0,$$
$$\frac{dz}{d\tau} = 0, \qquad z_0(s) = g(s).$$

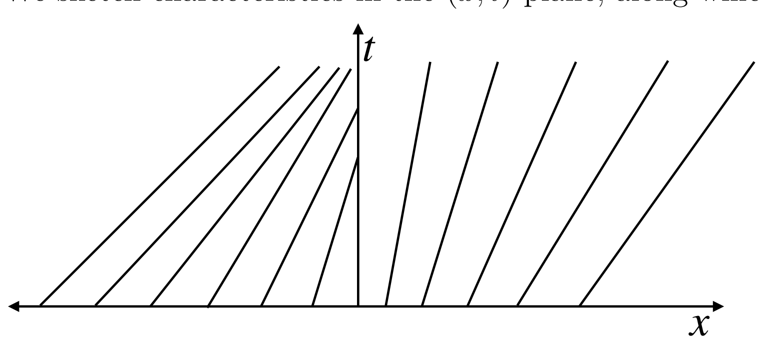
Locally, a unique solution exists since the Jacobian is nonzero along the initial curve

$$J = \begin{vmatrix} 1 & g(s)^2 \\ 0 & 1 \end{vmatrix} = 1.$$

Now, we can solve the ODE system sequentially: $t = \tau$; $z = g(s)$; so characteristics passing through $(x,t) = (s,0)$ are $x = g(s)^2 t + s$. Thus

$$u(x,t) = g(x - u^2 t).$$

We sketch characteristics in the (x,t) plane, along which $u = g(s)$ is constant.



(b) (10 points) Determine at what (x, t) a shock first forms in the solution from (a).

Solution: A shock occurs where u_x diverges, so differentiate the implicit solution

$$u_x = g'(s)(1 - 2uu_x t) \quad \Rightarrow \quad u_x = \frac{g'(s)}{1 + 2g(s)g'(s)t}.$$

The spatial derivative may thus diverge for $-2g(s)g'(s) > 0$ large enough. First, note $g(s) > 0$ and $g'(s) > 0$ for $s > 0$, so no shocks will occur in that region. However, when $s < 0$, we have $g(s) > 0$ and $g'(s) < 0$, so $-2g(s)g'(s) > 0$, so we need only check there for a maximum of $-g(s)g'(s) > 0$:

$$\frac{d}{ds}g(s)g'(s) = -\frac{d}{ds}\left[\mathrm{e}^s(1 - e^s)\right] = \mathrm{e}^s\left[2\mathrm{e}^s - 1\right] = 0 \quad \Rightarrow \quad s = -\log 2$$

is a critical point. Note the second derivative there is

$$-\frac{d}{ds}\mathrm{e}^s\left[2\mathrm{e}^s - 1\right]\Big|_{s=-\log 2} = -\frac{1}{2}[0] - \frac{1}{2}\cdot 2\frac{1}{2} = -\frac{1}{2} < 0,$$

so it is a max. Thus, the shock occurs at $x = -\log 2$ and

$$t = -\frac{1}{2g(-\log 2)g'(-\log 2)} = -\frac{1}{2(1/2)(-1/2)} = 2.$$

2. **Heat Equation.**

Consider the heat equation $\partial_t u = \partial_{xx} u$ in the strip $\Omega = \{(x,t) : 0 < x < L, t > 0\}$.

(a) (10 points) State the weak maximum and minimum principles for this equation with boundary conditions

$$u(x,0) = f(x), \qquad u(0,t) = g(t), \qquad u(L,t) = h(t).$$

Solution: Take any $T < \infty$ and $\bar{\Omega}_T \equiv \{(x,t) \in [0,L] \times [0,T]\}$ and $\Gamma_T \equiv \{(x,t)|(x = 0 \text{ or } L, t \in [0,T]) \cup (x \in [0,L], t = 0)\}$. The maximum principle states

$$\max_{(x,t)\in\bar{\Omega}_T} u(x,t) = \max_{(x,t)\in\Gamma_T} u(x,t) = \max\left[\max_{0\le x\le L} f(x), \max_{0\le t\le T} g(x), \max_{0\le t\le T} h(x)\right].$$

The minimum principle states that

$$\min_{(x,t)\in\Omega_T} u(x,t) = \min_{(x,t)\in\Gamma_T} u(x,t) = \min\left[\min_{0\le x\le L} f(x), \min_{0\le t\le T} g(x), \min_{0\le t\le T} h(x)\right].$$

(b) (15 points) Consider two solutions u_1 and u_2 to the same equation with two different boundary conditions:

$$\begin{aligned}
&\partial_t u_1 = \partial_{xx} u_1, \qquad u_1(x,0) = f_1(x), \qquad u_1(0,t) = g_1(t), \qquad u_1(L,t) = h_1(t),\\
&\partial_t u_2 = \partial_{xx} u_2, \qquad u_2(x,0) = f_2(x), \qquad u_2(0,t) = g_2(t), \qquad u_2(L,t) = h_2(t).
\end{aligned}$$

Prove that if $\max\limits_{0<x<L} |f_1(x) - f_2(x)| < \epsilon$, $\max\limits_{t>0} |g_1(t) - g_2(t)| < \epsilon$, $\max\limits_{t>0} |h_1(t) - h_2(t)| < \epsilon$, then $\max\limits_{(x,t)\in\bar{\Omega}} |u_1(x,t) - u_2(x,t)| < \epsilon$ for $\bar{\Omega} \equiv \{(x,t) : 0 \le x \le L, t \ge 0\}$.

Solution: The difference $w = u_1 - u_2$ satisfies

$$\begin{aligned}
&\partial_t w = \partial_{xx} w, \qquad w(x,0) = f_1(x) - f_2(x),\\
&w(0,t) = g_1(t) - g_2(t), \qquad w(L,t) = h_1(t) - h_2(t).
\end{aligned}$$

Applying the maximum and minimum principles from (a), we have

$$-\max_{x\in\Gamma_T} |w| \le -\max_{x\in\Gamma_T}(-w) = \min_{x\in\Gamma_T} w = \min_{x\in\bar{\Omega}_T} w \le \max_{x\in\bar{\Omega}_T} w = \max_{x\in\Gamma_T} w \le \max_{x\in\Gamma_T} |w|,$$

and so

$$\max_{x\in\bar{\Omega}_T} |u_1 - u_2| = \max_{x\in\bar{\Omega}_T} |w| \le \max_{x\in\Gamma_T} |w| < \epsilon$$

is true for any $T < \infty$, so $\max\limits_{(x,t)\in\Omega} |u_1(x,t) - u_2(x,t)| < \epsilon$.

3. **Wave Equation.** Consider the initial boundary value problem (IBVP) on the quarter plane:

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, & & c>0,\ c\neq 1, \quad x>0, \quad t>0, \\ u(x,0) &= \phi(x), \qquad u_t(x,0)=\psi(x), \quad x>0, & & \\ u_t(0,t) &= u_x(0,t), & & t>0, \quad \alpha \neq c, \end{aligned}$$

where $\phi, \psi \in C^2$ for $x > 0$ and $\lim_{x\to 0^+} \phi(x) = \lim_{x\to 0^+} \psi(x) = 0$.

(a) (15 points) Find the solution of the IBVP by first assuming $u(x,t) = F(x-ct) + G(x+ct)$ and using the initial and boundary conditions to specify $F(x)$ and $G(x)$ when $x > 0$ and $x < 0$.

Solution: Initial conditions imply that on $x > 0$:

$$\begin{aligned} \phi(x) &= F(x) + G(x), \text{ and } 0 = F(0) + G(0) \\ \psi(x) &= -cF'(x) + cG'(x), \text{ and } 0 = -cF'(0) + cG'(0) \\ \frac{1}{c}\int_0^x \psi(y)dy &= -F(x) + G(x) + A, \text{ and } 0 = -F(0) + G(0) + A \quad \to \quad A = 2F(0) \\ F(x) &= \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(y)dy + \frac{A}{2} \\ G(x) &= \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(y)dy - \frac{A}{2} \end{aligned}$$

so on $x > ct$,

$$u(x,t) = \frac{1}{2}\left[\phi(x-ct) + \phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy.$$

To obtain expressions for $F(x)$ and $G(x)$ when $x < 0$, use the boundary condition:

$$-cF'(-ct) + cG'(ct) = F'(-ct) + G'(ct)$$

and then integrate with respect to t to obtain

$$F(-ct) - F(0) + G(ct) - G(0) = -\frac{1}{c}\left[F(-ct) - F(0)\right] + \frac{1}{c}\left[G(ct) - G(0)\right]$$

and applying $F(0) + G(0) = 0$ and setting $x = -ct$, we have for $x < 0$

$$\begin{aligned} F(x) + G(-x) &= -\frac{1}{c}\left[F(x) - G(-x)\right] + \frac{2}{c}F(0) \\ F(x) &= -\frac{c-1}{c+1}G(-x) + \frac{1}{c+1}A \\ &= -\frac{c-1}{c+1}\left[\frac{\phi(-x)}{2} + \frac{1}{2c}\int_0^{-x}\psi(y)dy - \frac{A}{2}\right] + \frac{1}{c+1}A \\ &= -\frac{c-1}{c+1}\left[\frac{\phi(-x)}{2} + \frac{1}{2c}\int_0^{-x}\psi(y)dy\right] + \frac{A}{2} \end{aligned}$$

so that when $x < ct$, we have that

$$F(x-ct) = -\frac{c-1}{c+1}\left[\frac{\phi(ct-x)}{2} + \frac{1}{2c}\int_0^{ct-x}\psi(y)dy\right] + \frac{A}{2},$$

so when $0 < x < ct$, we have

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) - \frac{c-1}{c+1}\phi(ct-x)\right] - \frac{c-1}{c+1}\frac{1}{2c}\int_0^{ct-x}\psi(y)dy + \frac{1}{2c}\int_0^{x+ct}\psi(y)dy.$$

(b) (10 points) Use an energy argument to prove solutions to the IBVP are unique.

Solution: Define the energy functional for the wave equation $E(t) = \frac{1}{2}\int_0^\infty u_t^2 + u_x^2 dx$ and note that any two solutions u_1 and u_2 to the IBVP have difference $v = u_1 - u_2$ that satisfies

$$v_{tt} = c^2 v_{xx}, \quad x,t > 0; \quad v(x,0) \equiv v_t(x,0) \equiv 0, \quad x > 0; \quad v_t(0,t) = v_x(0,t), \quad t > 0.$$

By definition, any solution v to this IBVP satisfies $v_{tt} - c^2 v_{xx} \equiv 0$ for all $x > 0$ and $t \geq 0$, so for any $t \geq 0$

$$\begin{aligned} 0 &= \int_0^\infty v_t \cdot \left[v_{tt} - c^2 v_{xx}\right] dx = \frac{d}{dt}\int_0^\infty \frac{v_t^2}{2} dx + c^2 \int_0^\infty v_{xt} v_x dx - c^2 v_t v_x \Big|_0^\infty \\ &= \frac{d}{dt}\frac{1}{2}\int_0^\infty \left[v_t^2 + c^2 v_x^2\right] dx + c^2 v_x^2(0,t) = E'(t) + c^2 v_x^2(0,t) \quad \rightarrow E'(t) = -c^2 v_x^2(0,t), \end{aligned}$$

so $E'(t) \leq 0$ and $E(t) \geq 0$, since its integrand is nonnegative. Thus, since

$$E(0) = \frac{1}{2}\int_0^\infty 0^2 + c^2 0^2 dx = 0$$

and $E'(t) \leq 0$ and $E(t) \geq 0$ then $E(t) \equiv 0$, so $v_x \equiv v_t \equiv 0$ and since $v(x,0) \equiv 0$, then $v(x,t) \equiv 0$ for all $x > 0$ and $t \geq 0$, so $u_1 \equiv u_2$, so we can be sure solutions to the IBVP are unique.

4. **Poisson's Equation/Green's Functions.**

(a) (8 points) Consider the general Neumann problem for the Poisson equation:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^n,$$
$$-\frac{\partial u}{\partial n}(\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega.$$

Determine a condition relating $f(\mathbf{x})$ and $g(\mathbf{x})$ required for the boundary value problem (BVP) to have a solution.

Solution:

$$\int_\Omega f(\mathbf{x})d\mathbf{x} = -\int_\Omega \Delta u(\mathbf{x})d\mathbf{x} = -\int_{\partial\Omega} \frac{\partial u}{\partial n}(\mathbf{x})dS_x = \int_{\partial\Omega} g(\mathbf{x})dS_x.$$

(b) (7 points) The Green's function associated with the BVP in (a) satisfies the BVP

$$-\Delta G(\mathbf{x},\mathbf{y}) = \delta(\mathbf{x}-\mathbf{y}) + C, \qquad \mathbf{x},\mathbf{y} \in \Omega,$$
$$-\frac{\partial G}{\partial n}(\mathbf{x},\mathbf{y}) = 0, \qquad \mathbf{x} \in \partial\Omega, \quad \mathbf{y} \in \Omega.$$

Show that it follows that $C = -1/\int_\Omega d\mathbf{x}$.

Solution: Assuming $\mathbf{x},\mathbf{y} \in \Omega$, applying the divergence theorem shows:

$$C\int_\Omega d\mathbf{x} + 1 = \int_\Omega C + \delta(\mathbf{x}-\mathbf{y})d\mathbf{x} = -\int_\Omega \Delta G(\mathbf{x},\mathbf{y})d\mathbf{x} = -\int_{\partial\Omega} \frac{\partial G}{\partial n}(\mathbf{x})dS_x = 0 \quad \Rightarrow \quad C = -1/\int_\Omega d\mathbf{x}.$$

(c) (10 points) Use Green's second identity and the result from part (b) to derive the general solution to the BVP in (a) in terms of the Green's function. Specify the additive constant by requiring the mean of the solution over the domain Ω to be zero: $\bar{u} = \int_\Omega u(\mathbf{x})d\mathbf{x}/\int_\Omega d\mathbf{x} \equiv 0$.

Solution: Apply Green's Theorem to G and u to find

$$\int_\Omega (G\Delta u - u\Delta G)d\mathbf{x} = \int_{\partial\Omega}(G\frac{\partial u}{\partial n} - u\frac{\partial G}{\partial n})dS_x \quad \Rightarrow \quad \int_\Omega(-Gf + u(\delta(\mathbf{x}-\mathbf{y}) + C))d\mathbf{x} = \int_{\partial\Omega}(-Gg - u\cdot 0)dS_x$$
$$\Rightarrow \quad u(\mathbf{x}) = \int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y} - \int_{\partial\Omega} G(\mathbf{x},\mathbf{y})g(\mathbf{y})d\mathbf{y} + \bar{u} = \int_\Omega G(\mathbf{x},\mathbf{y})f(\mathbf{y})d\mathbf{y} - \int_{\partial\Omega} G(\mathbf{x},\mathbf{y})g(\mathbf{y})d\mathbf{y},$$

where the last step follows from requiring $\bar{u} \equiv 0$.

5. **Separation of Variables.**

Consider the initial value problem

$$a(x)\frac{\partial u}{\partial t} = +a(x)k(t)u + \frac{\partial}{\partial x}\left(b(x)\frac{\partial u}{\partial x}\right) + c(x)u, \qquad x \in (0,L),\ t > 0$$

with the boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad u(x,0) = f(x),$$

where $a(x), b(x) > 0$ for all x in $(0,L)$ and a, b, c, f are C^1 on $[0,L]$.

(a) (10 points) Write down a formal solution to the initial value problem in terms of the eigenfunctions $\phi_n(x)$ and eigenvalues λ_n of the associated Sturm-Liouville problem in the x variable. You can leave your answer in terms of ϕ_n, λ_n, and integrals of $k(t)$.

Solution: Assuming a solution of the form $u(x,t) = X(x)T(t)$ we obtain

$$aX\frac{\partial T}{\partial t} = akXT + T\frac{\partial}{\partial x}\left(b(x)\frac{\partial X}{\partial x}\right) + c(x)XT. \tag{1}$$

Dividing by aXT we get

$$\frac{1}{T}\frac{\partial T}{\partial t} - k(t) = \frac{1}{a(x)X}\frac{\partial}{\partial x}\left(b(x)\frac{\partial X}{\partial x}\right) + \frac{c(x)}{a(x)}. \tag{2}$$

Since the left hand side only depends on t and the right hand side only depends on x, they are equal to a constant $-\lambda$, so we get

$$\frac{\partial T}{\partial t} = [k(t) - \lambda]T, \tag{3}$$

$$\frac{\partial}{\partial x}\left(b(x)\frac{\partial X}{\partial x}\right) + c(x)X = -\lambda a(x)X. \tag{4}$$

Solving for $T(t)$ using separation of variables we get

$$T(t) = T(0)\exp\left(\int_0^t k(s)ds - \lambda t\right). \tag{5}$$

Eq. (4) with the boundary conditions $X(0) = 0$, $X(L) = 0$ is a regular, self-adjoint Sturm-Liouville problem, so there is a complete set of eigenfunctions $\phi_n(x)$ with eigenvalues $\lambda_1 < \lambda_2 < \ldots$, with $\lim_{n\to\infty} \lambda_n = \infty$. The general solution of the initial value problem is then given by a linear combination of the modes

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n \exp\left(\int_0^t k(s)ds - \lambda_n t\right)\phi_n(x). \tag{6}$$

To satisfy the initial conditions, we must choose the coefficients $\{\alpha_n\}$ so that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x). \tag{7}$$

Since the eigenfunctions $\phi_n(x)$ are orthogonal under the inner product $\langle u, v\rangle = \int_0^L a(x)u(x)v(x)dx$, one finds

$$\alpha_n = \frac{\int_0^L a(x)f(x)\phi_n(x)dx}{\int_0^L a(x)\phi_n(x)^2 dx}. \tag{8}$$

(b) (15 points) Show that if $\int_0^\infty k(s)ds < \infty$ and $c(x) < 0$ for all x in $(0, L)$, then $\lim_{t\to\infty} \|u\|^2 = 0$, where $\|\cdot\|$ is the norm associated with the Sturm-Liouville inner product. Hint: Show the eigenvalues are positive using the Rayleigh quotient $\langle \mathcal{L}\phi_n, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$.

Solution: To understand the behavior of the solution as $t \to \infty$, we need to study the eigenvalues. Using the Rayleigh quotient, we have

$$\begin{aligned}
\lambda_n &= \lambda_n \frac{\langle \phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} = \frac{\langle \mathcal{L}\phi_n, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle} \\
&= \frac{-b(x)\phi_n(x)\phi_n'(x)|_0^L + \int_0^L [b(x)|\phi_n'(x)|^2 - c(x)|\phi_n(x)|^2]dx}{\int_0^L a(x)|\phi_n(x)|^2]dx} \\
&= \frac{\int_0^L [b(x)|\phi_n'(x)|^2 - c(x)|\phi_n(x)|^2]dx}{\int_0^L a(x)|\phi_n(x)|^2]dx},
\end{aligned}$$

which is positive if $c(x) < 0$ in $(0, L)$, i.e., $\lambda_n \geq 0$.

We have

$$\begin{aligned}
\|u\|^2 &= \langle u(x,t), u(x,t) \rangle \\
&= \Big\langle \sum_{n=1}^{\infty} \alpha_n \exp\left(\int_0^t k(s)ds - \lambda_n t\right) \phi_n(x), \sum_{m=1}^{\infty} \alpha_m \exp\left(\int_0^t k(s)ds - \lambda_m t\right) \phi_m(x) \Big\rangle \\
&= \sum_{n=1}^{\infty} \alpha_n^2 \|\phi_n\|^2 \exp\left(2\int_0^t k(s)ds - 2\lambda_n t\right) \\
&\leq \exp\left(2\int_0^t k(s)ds - 2\lambda_1 t\right) \sum_{n=1}^{\infty} \alpha_n^2 \|\phi_n\|^2 \\
&= \exp\left(2\int_0^t k(s)ds - 2\lambda_1 t\right) \|f\|^2.
\end{aligned}$$

Since a and f are C^1 on $[0, L]$, then $\|f\|^2 = \int_0^L a(x)f^2(x)dx < \infty$. If $\int_0^\infty k(s)ds < \infty$ then, since $\lambda_1 > 0$, the exponential term has zero limit as $t \to \infty$ and $\lim_{t\to\infty} \|u\|^2 = 0$.