Preliminary Exam
Partial Differential Equations
9:00 AM - 12:00 PM, Aug. 20, 2024
Newton Lab, ECCR 257

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of formulae is provided.

1. **Method of characteristics** Two of the following three problems cannot be solved as stated:

   (a) $\partial_x u + \partial_y u = u^2$ with initial data $x = s$, $y = -s$, $u = s$, $s \in \mathbb{R}$.

   (b) $\partial_x u + \partial_y u = u$ with initial data $x = s$, $y = s$, $u = 1$, $s \in \mathbb{R}$.

   (c) $x\partial_x u + y\partial_y u = u$ with initial data $x = s$, $y = -s$, $u = s$, $s \in \mathbb{R}$.

   (7 points) Identify the unsolvable problems, and explain why they are unsolvable.

   For the remaining problem:

   (i) (3 points) Do the characteristics cross? If so, where?

   (ii) (5 points) Find the solution and evaluate it (i.e., give a numerical value) at $(x, y) = (2, 3)$.

   (iii) (5 points) The solution of this problem is singular somewhere in the (x, y) plane (including possibly at infinity). Where is it singular? What is the nature of the singularity (e.g., $|u| \to \infty$, $|\partial_x u| \to \infty$, etc)?

   (iv) (5 points) Sketch the characteristics, the curve where initial data is specified, and the curve where the solution is singular in the (x, y) plane.

   Solution: For (b), there is only one characteristic curve, $x(\tau) = \tau + s$ and $y(\tau) = \tau + s$ for each $s \in \mathbb{R}$, i.e., $x = y$. The curve where initial data is specified is the same, $x = y$. Therefore, it is not possible to flow off of the initial data curve and the Inverse Function Theorem can't be applied to find (x, y) as a function of (s, τ) and the method of characteristics fails. For (c), again there is just one characteristic curve, $x(\tau) = se^\tau$ and $y(\tau) = -se^\tau$ for each $s \in \mathbb{R}$, i.e., $x = -y$, which also coincides with the initial data curve. Thus (b) and (c) do not satisfy the transversality condition on the initial data so are therefore unsolvable.

   For (a), we set up the equations

$$\frac{dx}{d\tau} = 1, \qquad x(0, s) = s, \tag{1}$$

$$\frac{dy}{d\tau} = 1, \qquad y(0, s) = -s, \tag{2}$$

$$\frac{dz}{d\tau} = z^2, \qquad z(0, s) = s, \tag{3}$$

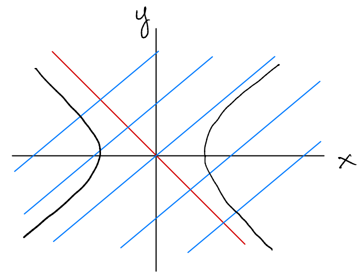

Figure 1: Characteristic plane for problem 1a with initial data (red), characteristic curves (blue), and the singular curve (black).

with solution

$$x(\tau, s) = \tau + s, \quad (4)$$
$$y(\tau, s) = \tau - s, \quad (5)$$
$$z(\tau, s) = \frac{s}{1 - \tau s}. \quad (6)$$

(i) The characteristics are given by $y = x - 2s$, so they never cross.

(ii) We have $\tau = (x + y)/2$ and $s = (x - y)/2$, so

$$u(x, y) = z(\tau(x, y), s(x, y)) = \frac{(x - y)/2}{1 - (x - y)(x + y)/4} = \frac{2(x - y)}{4 - (x^2 - y^2)}.$$

Then, $u(2, 3) = -\frac{2}{9}$.

(iii) The solution is singular ($|u| \to \infty$) on the hyperbola $x^2 - y^2 = 4$.

(iv) The initial data (red), characteristic curves (blue), and the singular curve (black) are shown in Fig. 1.

2. **Heat Equation** Consider Green's function $G(x, t)$ satisfying

$$G_t = G_{xx} \quad -\infty < x < \infty, \quad t > 0, \quad (7)$$
$$G(x, 0) = \delta(x), \quad (8)$$

where $\delta(x)$ is the Dirac delta distribution.

(a) (9 points) Use Fourier transforms to establish that

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - k^2 t} dk.$$

Solution: Let

$$\hat{G}(k, t) = \int_{-\infty}^{\infty} e^{-ikx} G(x, t) dx$$

be the Fourier transform of $G(x, t)$, so that

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{G}(k, t) e^{ikx} dk.$$

Taking the Fourier transform of $G_t = G_{xx}$ and using properties of Fourier transforms,

$$\hat{G}_t = -k^2 \hat{G}.$$

Integrating this ODE in t yields

$$\hat{G}(k,t) = \hat{G}(k,0)e^{-k^2 t}.$$

To obtain the initial condition, we compute the Fourier transform of the delta distribution

$$\hat{\delta}(k) = (\delta(\cdot), e^{-ik\cdot}) = \int_{-\infty}^{\infty} e^{-ikx}\delta(x)dx = 1 \quad \Rightarrow \quad \hat{G}(k,0) = 1.$$

Then $\hat{G}(k,t) = e^{-k^2 t}$. Taking the inverse Fourier transform provides the desired result

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{G}(k,t)e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-k^2 t}dk.$$

(b) (9 points) Show that the above integral can be evaluated in closed form and find

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right).$$

Solution: Completing the square

$$ikx - k^2 t = -t\left(k - \frac{ix}{2t}\right)^2 - \frac{x^2}{4t},$$

the integral can be expressed

$$G(x,t) = \frac{1}{2\pi}e^{-x^2/(4t)}\int_{-\infty}^{\infty} e^{-t(k-ix/(2t))^2}dk.$$

Making the change of variable $y = \sqrt{t}(k - ix/(2t))$, then $dk = dy/\sqrt{t}$ and

$$G(x,t) = \frac{1}{2\pi}e^{-x^2/(4t)}\int_{-\infty}^{\infty} e^{-y^2}dy/\sqrt{t}.$$

To integrate the Gaussian, we use the polar coordinate trick

$$\int_{-\infty}^{\infty} e^{-y^2}dy = \left(\int_{-\infty}^{\infty} e^{-y_1^2}dy_1 \int_{-\infty}^{\infty} e^{-y_2^2}dy_2\right)^{1/2} = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(y_1^2+y_2^2)}dy_1 dy_2\right)^{1/2} \tag{9}$$

$$= \left(\int_0^{2\pi}\int_0^{\infty} e^{-r^2} r dr d\theta\right)^{1/2} = \left(\int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\Big|_{r=0}^{\infty}\right)d\theta\right)^{1/2} \tag{10}$$

$$= \sqrt{\pi}. \tag{11}$$

Putting this together, we obtain the fundamental solution of the heat equation or heat kernel

$$G(x,t) = \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right).$$

(c) (3 points) Use Green's function to construct the solution to the initial value problem

$$u_t = u_{xx} \quad -\infty < x < \infty, \;\; t > 0 \tag{12}$$

$$u(x,0) = h(x), \tag{13}$$

Solution: Use the convolution in x

$$u(x,t) = G * h = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} h(y)e^{-(x-y)^2/(4t)}dy.$$

(d) (4 points) Suppose the non-negative, continuous function $h(x)$ has compact support and $h(0) = 1$, i.e., there is $L > 0$ such that $h(x) = 0$ for $|x| > L$. Thus $u(2L, 0) = 0$. Find the smallest time such that $u(2L, t) \neq 0$.

Solution: There is no earliest time. By continuity of h, there exists $\delta > 0$ such that $h(y) > 1/2$ for all $|y| < \delta$. Then,

$$u(2L,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} h(y) e^{-(2L-y)^2/(4t)} dy \tag{14}$$

$$\geq \frac{1}{\sqrt{4\pi t}} \int_{-\delta}^{\delta} h(y) e^{-(2L-y)^2/(4t)} dy \tag{15}$$

$$> \frac{1}{\sqrt{4\pi t}} \int_{-\delta}^{\delta} \frac{1}{2} e^{-(2L+y)^2/(4t)} dy \tag{16}$$

$$> \frac{\delta}{\sqrt{4\pi t}} e^{-(2L+\delta)^2/(4t)} dy > 0, \tag{17}$$

for all $t > 0$. This represents infinite speed of propagation.

3. **Wave Equation**

Consider

$$\begin{aligned} &u_{tt} - c^2 u_{xx} + a u_t + \frac{a^2}{4} u = 0, \qquad 0 \leq x \leq L, \quad t > 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad u(0,t) = u(L,t) = 0, \end{aligned} \tag{18}$$

where $f(x)$, $g(x)$ are integrable and $c > 0$ and $a > 0$ are real constants.

(a) (15 points)

Obtain a formal series solution to the above initial boundary value problem.

Solution: Substituting $u(x,t) = e^{-\frac{a}{2}t} w(x,t)$ into (18) gives $w_{tt} - c^2 w_{xx} = 0$. Using separation of variables $w(x,t) = X(x)T(t)$, gives $T''(t)/T(t) = c^2 X''(x)/X(x) = -k^2$, where $k^2 \geq 0$. Then, standard methods give the formal solution

$$u(x,t) = e^{-\frac{a}{2}t} \sum_{n=1}^{\infty} \sin\left(\frac{\pi n x}{L}\right) \left[A_n \cos\left(\frac{\pi n c t}{L}\right) + B_n \sin\left(\frac{\pi n c t}{L}\right)\right]$$

The Fourier coefficients are defined by

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{\pi n x}{L}\right) dx$$

and

$$B_n = \frac{L}{\pi n c} \left(\frac{2}{L} \int_0^L \left[g(x) + \frac{a}{2} f(x)\right] \sin\left(\frac{\pi n x}{L}\right) dx \right)$$

(b) (5 points)

Derive the energy relation

$$\begin{aligned} \frac{dE}{dt} &= -2a \int_0^L u_t^2 dx, \\ E(t) &= \int_0^L \left[u_t^2 + c^2 u_x^2 + \frac{a^2}{4} u^2\right] dx \end{aligned} \tag{19}$$

What physical effect do the additional terms au_t and $a^2u/4$ in (18) represent?

Solution: Multiply (18) by u_t and integrate over the interval $[0, L]$. This gives

$$\int_0^L \left(\frac{1}{2} \left(u_t^2\right)_t - c^2 u_t u_{xx} + a u_t^2 + \frac{a^2}{8}(u^2)_t \right) dx = 0.$$

The boundary conditions $u(0, t) = u(L, t) = 0$ imply $u_t(0, t) = u_t(L, t) = 0$. Performing integration-by-parts on the second term and applying these boundary conditions yields the desired energy relation

$$\frac{1}{2}\frac{d}{dt} \int_0^L \left(u_t^2 + c^2 u_x^2 + \frac{a^2}{4} u^2 \right) dx = -a \int_0^L u_t^2 dx\,.$$

The non-negative definite energy $E(t)$ is non-increasing in time, i.e. $E(t_2) \leq E(t_1)$ for $t_2 > t_1$, indicating some dissipative force (e.g. friction, vibration) is modeled by the terms au_t and $a^2u/4$.

(c) (5 points)
Using the energy relation (19), prove that the solution found in part (a) is unique.

Solution: Suppose (18) has two distinct solutions: $u_1(x, t)$ and $u_2(x, t)$. Define $\tilde{u} \equiv u_1 - u_2$, which satisfies the equation $\tilde{u}_{tt} - c^2\tilde{u}_{xx} + a\tilde{u}_t + \frac{a^2}{4}\tilde{u} = 0$ and initial conditions $\tilde{u}(x, 0) = 0$, $\tilde{u}_x(x, 0) = 0$, $\tilde{u}_t(x, 0) = 0$. As a result, the energy relation (19) satisfies $0 \leq E(t) \leq E(0) = 0$. This implies that $E(t) = 0$ for all $t > 0$, or $E(t) = \int_0^L \left[\tilde{u}_t^2 + u_x^2 + \frac{a^2}{4}\tilde{u}^2\right] dx = 0$. Since $\tilde{u}$ is smooth, this means that $\tilde{u}_t^2 + \tilde{u}_x^2 + \frac{a^2}{4}\tilde{u}^2 = 0$. Since these are all non-negative quantities this implies that $\tilde{u}_t = \tilde{u}_x = \tilde{u} = 0$, or equivalently $u_1(x, t) = u_2(x, t)$.

4. **Fourier Series and Convergence** Let $f(x)$ be a piecewise smooth, $2L$-periodic function. Let a_n and b_n be the Fourier coefficients corresponding to the cosine and sine terms, respectively of f and α_n and β_n be the Fourier coefficients corresponding to the cosine and sine terms, respectively of f'.

(a) (15 points) Prove that a_n is $\mathcal{O}(n^{-1})$.

Solution: The Fourier coefficient for the cos term is defined as

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos({}^{n\pi x}/_{L}) \mathrm{d}x\,.$$

Since $f(x)$ is piecewise smooth, by defintion, $f(x)$ and $f'(x)$ are piecewise continuous. Thus integration by parts yields

$$a_n = \underbrace{\frac{f(x)}{n\pi} \sin({}^{n\pi x}/_{L}) \bigg|_{-L}^{L}}_{=0} - \frac{1}{n\pi} \int_{-L}^{L} f'(x) \sin({}^{n\pi x}/_{L}) \mathrm{d}x\,.$$

Since f' is piecewise continuous, we can conclude that $|f'| \leq M$ for $0 \leq M \in \mathbb{R}$ and thus

$$|a_n| \leq \frac{M}{n\pi} \int_{-L}^{L} \mathrm{d}x = \frac{2LM}{n\pi} = \mathcal{O}({}^{1}/_{n})\,.$$

(b) (10 points) If $\lim_{x\searrow -L} f(x) = \lim_{x\nearrow L} f(x)$, then prove that $a_n \to 0$ faster than $\mathcal{O}(n^{-1})$, i.e., $a_n = o(n^{-1})$.

Solution:

Note that the Fourier coefficient of the sin term for f' is

$$\beta_n = \frac{1}{L}\int_{-L}^{L} f'(x)\sin({}^{n\pi x}/_{L})\mathrm{d}x\,.$$

Consider the fact that f' is piecewise continuous on $[-L, L]$ except possibly for a finite number of points. Thus we can conclude that $\int_{-L}^{L} |f'(x)|\mathrm{d}x < \infty$ and therefore the Riemann-Lebesgue Lemma applies, which yields

$$\lim_{n\to\infty}\int_{-L}^{L} f'(x)\sin({}^{n\pi x}/_{L})\mathrm{d}x = 0\,.$$

From part a), we can observe that $a_n = -({}^{L}/_{n\pi})\beta_n$ and thus $a_n = o({}^{1}/_{n})$.

5. **Separation of Variables** Consider the initial boundary value problem

$$u_t = 4u_{xx} + e^{-2t}, \quad 0 < x < 1,\ t > 0, \tag{20}$$
$$u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \tag{21}$$
$$u(x,0) = \phi(x), \quad 0 < x < 1. \tag{22}$$

(a) (6 points) Interpret each one of the equations and conditions above in terms of heat flow.

Solution: The PDE models Fourier's law of heat conduction—$u(x,t)$ is the temperature at location x and time t—along a one-dimensional rod of unit length with (non-dimensional) thermal conductivity 4 subject to a spatially independent cooling e^{-2t}, e.g., a cooling bath. The boundary conditions correspond to no heat flux through the boundaries, i.e., the rod ends are insulated. The initial condition corresponds to the initial temperature distribution.

(b) (12 points) Use separation of variables to construct a formal series solution. Assuming convergence of the series, what is the limit $\lim_{t\to\infty} u(x,t)$?

Solution: Let $u(x,t) = -\frac{1}{2}e^{-2t} + w(x,t)$, then w satisfies the homogeneous heat equation $w_t = 4w_{xx}$ subject to $w_x(0,t) = w_x(1,t) = 0$ and $w(x,0) = \phi(x) + \frac{1}{2}$. Separated solutions are $w_n(x,t) = \cos(n\pi x)e^{-(2n\pi)^2 t}$, $n = 0, 1, 2, \ldots$. Introducing the formal series solution

$$w(x,t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x)e^{-(2n\pi)^2 t},$$

we use $w(x,0) = \frac{1}{2} + \phi(x)$ to obtain the Fourier coefficients

$$a_0 = \frac{1}{2} + \int_0^1 \phi(x)dx, \quad a_n = 2\int_0^1 \phi(x)\cos(n\pi x)dx, \quad n = 1, 2, \ldots.$$

Then, the formal series solution for $u(x,t)$ is

$$u(x,t) = \frac{1}{2}(1 - e^{-2t}) + \int_0^1 \phi(y)dy + \sum_{n=1}^{\infty} a_n \cos(n\pi x)e^{-(2n\pi)^2 t}.$$

Evaluating the limit term-by-term, we obtain

$$\lim_{t\to\infty} u(x,t) = a_0 = \frac{1}{2} + \int_0^1 \phi(x)dx.$$

(c) (7 points) Determine sufficient non-trivial conditions on $\phi(x)$ so that the formal solution is a classical solution and prove it.

Solution: If $\phi(x)$ is bounded and integrable on $0 \leq x \leq 1$, say $|\phi(x)| < M$. Then, $|a_n| \leq 2M$ and the series can be differentiated term by term. For example, the terms obtained by term-by-term differentiation $u_x x(x,t)$ are bounded as

$$|(n\pi)^2 a_n \cos(n\pi x) e^{-(2n\pi)^2 t}| \leq 2M(n\pi)^2 e^{-(2n\pi)^2 t},$$

so that the series for $u_{xx}(x,t)$ converges by the ratio test and the Weierstrass M-test guarantees its uniform convergence when $t > 0$ and $x \in [0,1]$. Thus, the PDE is satisfied for $t > 0$ and $x \in [0,1]$. Similarly, term by term differentiation compels $u_x(0,t) = u_x(1,t) = 0$ for $t > 0$. In order to guarantee that the series satisfies the initial condition, we further assume that $\phi''(x)$ is continuous on $[0,1]$ so that performing integration by parts twice we have

$$a_n = \frac{2}{(n\pi)^2}((-1)^n \phi'(1) - \phi'(0)) - \frac{2}{(n\pi)^2} \int_0^1 \phi''(x) \cos(n\pi x) dx,$$

so that $|a_n| \leq 4M'/(n\pi)^2 + 2M''/(n\pi)^2$ for $|\phi'(x)| \leq M'$ and $|\phi''(x)| \leq M''$. Consequently $\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} C/n^2 < \infty$ where $C = 2(2M' + M'')/\pi^2$ and the series for $u(x,0)$ converges to the initial condition uniformly by the ratio and Weierstrass M-test for $x \in (0,1)$.

♢	♠	**END**	♡	♣