Preliminary Exam
Partial Differential Equations
9:00 AM - 12:00 PM, Aug. 20, 2024
Newton Lab, ECCR 257

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points. A sheet of formulae is provided.

1. **Method of characteristics** Two of the following three problems cannot be solved as stated:

   (a) $\partial_x u + \partial_y u = u^2$ with initial data $x = s$, $y = -s$, $u = s$, $s \in \mathbb{R}$.

   (b) $\partial_x u + \partial_y u = u$ with initial data $x = s$, $y = s$, $u = 1$, $s \in \mathbb{R}$.

   (c) $x\partial_x u + y\partial_y u = u$ with initial data $x = s$, $y = -s$, $u = s$, $s \in \mathbb{R}$.

   (7 points) Identify the unsolvable problems, and explain why they are unsolvable.

   For the remaining problem:

   (i) (3 points) Do the characteristics cross? If so, where?

   (ii) (5 points) Find the solution and evaluate it (i.e., give a numerical value) at $(x, y) = (2, 3)$.

   (iii) (5 points) The solution of this problem is singular somewhere in the (x, y) plane (including possibly at infinity). Where is it singular? What is the nature of the singularity (e.g., $|u| \to \infty$, $|\partial_x u| \to \infty$, etc)?

   (iv) (5 points) Sketch the characteristics, the curve where initial data is specified, and the curve where the solution is singular in the (x, y) plane.

2. **Heat Equation** Consider Green's function $G(x, t)$ satisfying

$$G_t = G_{xx} \quad -\infty < x < \infty, \;\; t > 0, \tag{1}$$
$$G(x, 0) = \delta(x), \tag{2}$$

   where $\delta(x)$ is the Dirac delta distribution.

   (a) (9 points) Use Fourier transforms to establish that

$$G(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx - k^2 t} dk.$$

   (b) (9 points) Show that the above integral can be evaluated in closed form and find

$$G(x, t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right).$$

   (c) (3 points) Use Green's function to construct the solution to the initial value problem

$$u_t = u_{xx} \quad -\infty < x < \infty, \;\; t > 0 \tag{3}$$
$$u(x, 0) = h(x), \tag{4}$$

(d) (4 points) Suppose the non-negative, continuous function $h(x)$ has compact support and $h(0) = 1$, i.e., there is $L > 0$ such that $h(x) = 0$ for $|x| > L$. Thus $u(2L, 0) = 0$. Find the smallest time such that $u(2L, t) \neq 0$.

3. **Wave Equation**
   Consider

$$u_{tt} - c^2 u_{xx} + a u_t + \frac{a^2}{4} u = 0, \qquad 0 \le x \le L, \quad t > 0, \tag{5}$$
$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \qquad u(0,t) = u(L,t) = 0,$$

where $f(x)$, $g(x)$ are integrable and $c > 0$ and $a > 0$ are real constants.

(a) (15 points)
Obtain a formal series solution to the above initial boundary value problem.

(b) (5 points)
Derive the energy relation

$$\frac{dE}{dt} = -2a \int_0^L u_t^2 dx, \tag{6}$$
$$E(t) = \int_0^L \left[u_t^2 + c^2 u_x^2 + \frac{a^2}{4} u^2 \right] dx$$

What physical effect do the additional terms $a u_t$ and $a^2 u/4$ in (5) represent?

(c) (5 points)
Using the energy relation (6), prove that the solution found in part (a) is unique.

4. **Fourier Series and Convergence** Let $f(x)$ be a piecewise smooth, $2L$-periodic function. Let a_n and b_n be the Fourier coefficients corresponding to the cosine and sine terms, respectively of f and α_n and β_n be the Fourier coefficients corresponding to the cosine and sine terms, respectively of f'.

(a) (15 points) Prove that a_n is $\mathcal{O}(n^{-1})$.

(b) (10 points) If $\lim_{x \searrow -L} f(x) = \lim_{x \nearrow L} f(x)$, then prove that $a_n \to 0$ faster than $\mathcal{O}(n^{-1})$, i.e., $a_n = o(n^{-1})$.

5. **Separation of Variables** Consider the initial boundary value problem

$$u_t = 4u_{xx} + e^{-2t}, \quad 0 < x < 1, \ t > 0, \tag{7}$$
$$u_x(0,t) = u_x(1,t) = 0, \quad t > 0, \tag{8}$$
$$u(x,0) = \phi(x), \quad 0 < x < 1. \tag{9}$$

(a) (6 points) Interpret each one of the equations and conditions above in terms of heat flow.

(b) (12 points) Use separation of variables to construct a formal series solution. Assuming convergence of the series, what is the limit $\lim_{t\to\infty} u(x,t)$?

(c) (7 points) Determine sufficient non-trivial conditions on $\phi(x)$ so that the formal solution is a classical solution and prove it.

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