Preliminary Exam
Partial Differential Equations
1:00 - 4:00 PM, Wednesday, Jan. 5, 2022
Remotely

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.**

   (a) (15 points) Obtain an implicit solution to the following initial value problem (IVP):

   $$u_t + u^2 u_x = 0, \qquad x \in (-\infty, \infty), \quad t > 0,$$
   $$u(x, 0) = g(x) = 1 - \mathrm{e}^{-|x|}.$$

   Remember to check existence and uniqueness near the initial condition. Also, sketch the characteristics of the associated nonlinear wave solution in the (x, t) plane.

   (b) (10 points) Determine at what (x, t) a shock first forms in the solution from (a).

2. **Heat Equation.**

   Consider the heat equation $\partial_t u = \partial_{xx} u$ in the strip $\Omega = \{(x,t) : 0 < x < L, t > 0\}$.

   (a) (10 points) State the weak maximum and minimum principles for this equation with boundary conditions

   $$u(x,0) = f(x), \qquad u(0,t) = g(t), \qquad u(L,t) = h(t).$$

   (b) (15 points) Consider two solutions u_1 and u_2 to the same equation with two different boundary conditions:

   $$\begin{aligned} &\partial_t u_1 = \partial_{xx} u_1, \qquad u_1(x,0) = f_1(x), \qquad u_1(0,t) = g_1(t), \qquad u_1(L,t) = h_1(t), \\ &\partial_t u_2 = \partial_{xx} u_2, \qquad u_2(x,0) = f_2(x), \qquad u_2(0,t) = g_2(t), \qquad u_2(L,t) = h_2(t). \end{aligned}$$

   Prove that if $\max\limits_{0<x<L} |f_1(x) - f_2(x)| < \epsilon$, $\max\limits_{t>0} |g_1(t) - g_2(t)| < \epsilon$, $\max\limits_{t>0} |h_1(t) - h_2(t)| < \epsilon$, then $\max\limits_{(x,t)\in\bar{\Omega}} |u_1(x,t) - u_2(x,t)| < \epsilon$ for $\bar{\Omega} \equiv \{(x,t) : 0 \le x \le L, t \ge 0\}$.

3. **Wave Equation.** Consider the initial boundary value problem (IBVP) on the quarter plane:

   $$\begin{aligned} u_{tt} &= c^2 u_{xx}, && c > 0,\ c \neq 1, \quad x > 0, \quad t > 0, \\ u(x,0) &= \phi(x), \qquad u_t(x,0) = \psi(x), && x > 0, \\ u_t(0,t) &= u_x(0,t), && t > 0, \quad \alpha \neq c, \end{aligned}$$

   where $\phi, \psi \in C^2$ for $x > 0$ and $\lim_{x\to 0^+} \phi(x) = \lim_{x \to 0^+} \psi(x) = 0$.

   (a) (15 points) Find the solution of the IBVP by first assuming $u(x,t) = F(x-ct) + G(x+ct)$ and using the initial and boundary conditions to specify $F(x)$ and $G(x)$ when $x > 0$ and $x < 0$.

   (b) (10 points) Use an energy argument to prove solutions to the IBVP are unique.

4. **Poisson's Equation/Green's Functions.**

(a) (8 points) Consider the general Neumann problem for the Poisson equation:

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \subset \mathbb{R}^n,$$
$$-\frac{\partial u}{\partial n}(\mathbf{x}) = g(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega.$$

Determine a condition relating $f(\mathbf{x})$ and $g(\mathbf{x})$ required for the boundary value problem (BVP) to have a solution.

(b) (7 points) The Green's function associated with the BVP in (a) satisfies the BVP

$$-\Delta G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) + C, \qquad \mathbf{x}, \mathbf{y} \in \Omega,$$
$$-\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) = 0, \qquad \mathbf{x} \in \partial\Omega, \quad \mathbf{y} \in \Omega.$$

Show that it follows that $C = -1/\int_\Omega d\mathbf{x}$.

(c) (10 points) Use Green's second identity and the result from part (b) to derive the general solution to the BVP in (a) in terms of the Green's function. Specify the additive constant by requiring the mean of the solution over the domain Ω to be zero: $\bar{u} = \int_\Omega u(\mathbf{x})d\mathbf{x}/\int_\Omega d\mathbf{x} \equiv 0$.

5. **Separation of Variables.**

Consider the initial value problem

$$a(x)\frac{\partial u}{\partial t} = +a(x)k(t)u + \frac{\partial}{\partial x}\left(b(x)\frac{\partial u}{\partial x}\right) + c(x)u, \qquad x \in (0, L),\ t > 0$$

with the boundary conditions

$$u(0, t) = u(L, t) = 0, \qquad u(x, 0) = f(x),$$

where $a(x), b(x) > 0$ for all x in $(0, L)$ and a, b, c, f are C^1 on $[0, L]$.

(a) (10 points) Write down a formal solution to the initial value problem in terms of the eigenfunctions $\phi_n(x)$ and eigenvalues λ_n of the associated Sturm-Liouville problem in the x variable. You can leave your answer in terms of ϕ_n, λ_n, and integrals of $k(t)$.

(b) (15 points) Show that if $\int_0^\infty k(s)ds < \infty$ and $c(x) < 0$ for all x in $(0, L)$, then $\lim_{t\to\infty} \|u\|^2 = 0$, where $\|\cdot\|$ is the norm associated with the Sturm-Liouville inner product. Hint: Show the eigenvalues are positive using the Rayleigh quotient $\langle \mathcal{L}\phi_n, \phi_n\rangle / \langle \phi_n, \phi_n\rangle$.