Preliminary Exam
Partial Differential Equations
9:00 AM - 12:00 PM, Jan. 11, 2024
Newton Lab, ECCR 257

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.

A sheet of convenient formulae is provided.

1. **Method of characteristics.** Consider the inviscid Burger's equation

$$u_t + uu_x = 0 \tag{1}$$

on the domain $\Omega = \mathbb{R} \times \mathbb{R}^+$ with initial conditions

$$u(x,0) = u_0(x) = \begin{cases} 1, & x \le 0, \\ 1 - x, & 0 < x \le 1, \\ 0, & 1 < x. \end{cases} \tag{2}$$

(a) Find the time and position at which a shock forms.
Solution: The characteristic equations are

$$\frac{dt}{d\tau} = 1, \tag{3}$$
$$\frac{dx}{d\tau} = u, \tag{4}$$
$$\frac{du}{d\tau} = 0, \tag{5}$$
$$\tag{6}$$

which gives, using the initial data $(x, t, u) = (s, 0, u_0(s))$,

$$t = \tau, \tag{7}$$
$$x = ut + s, \tag{8}$$
$$u = u_0(s). \tag{9}$$

Thus, the solution u satisfies the implicit equation $u = u_0(x - ut)$. To find the location of the shock, we differentiate with respect to x and solve for u_x, finding

$$u_x = \frac{u_0'}{1 + u_0' t}. \tag{10}$$

Thus, a characteristic emanating from the initial point x_0 will lead to a diverging derivative when $u_0'(x_0)t = -1$. Since

$$u_0'(x) = \begin{cases} 0, & x \le 0, \\ -1, & 0 < x \le 1, \\ 0, & 1 < x, \end{cases} \tag{11}$$

we conclude that all characteristics emanating from $(0,1)$ produce a shock at $t_s = 1$. The position of the shock for the characteristic starting at $x_0 = s \in (-1,1)$ can be found by setting $t = t_s = 1$ and $u = u_0(s) = 1 - s$ in Eq. (8), which gives $x_s = (1-s)1 + s = 1$. Therefore the shock forms at $(x_s, t_s) = (1,1)$.

(b) Find the subsequent trajectory of the discontinuous shock by applying the Rankine-Hugoniot condition

$$s(t) = \frac{1}{2}(u_-(t) + u_+(t)),$$

where s is the speed of the discontinuity and $u_\pm(t) = \lim_{x \to x_s(t)^\pm} u(x,t)$ and $s = \dot{x}_s(t)$.

Solution: Since the Burgers equation can be written as $u_t + (u^2/2)_x = 0$, the Rankine-Hugoniot condition for the position of the shock $x_s(t)$ gives

$$\frac{dx_s}{dt} = \frac{\frac{1}{2}u_+^2 - \frac{1}{2}u_-^2}{u_+ - u_-}, \tag{12}$$

where u_+ and u_- are the values of u to the right and to the left of the shock, respectively. The value to the left corresponds to characteristics emanating from $x_0 < 0$, for which $u = 1$, and the value to the left corresponds to characteristics emanating from $x_0 > 1$, for which $u = 0$ (a rough sketch of the characteristics might be useful here). Thus, $u_+ = 0$ and $u_- = 1$, and we have

$$\frac{dx_s}{dt} = \frac{\frac{1}{2}0 - \frac{1}{2}1}{0 - 1} = \frac{1}{2}. \tag{13}$$

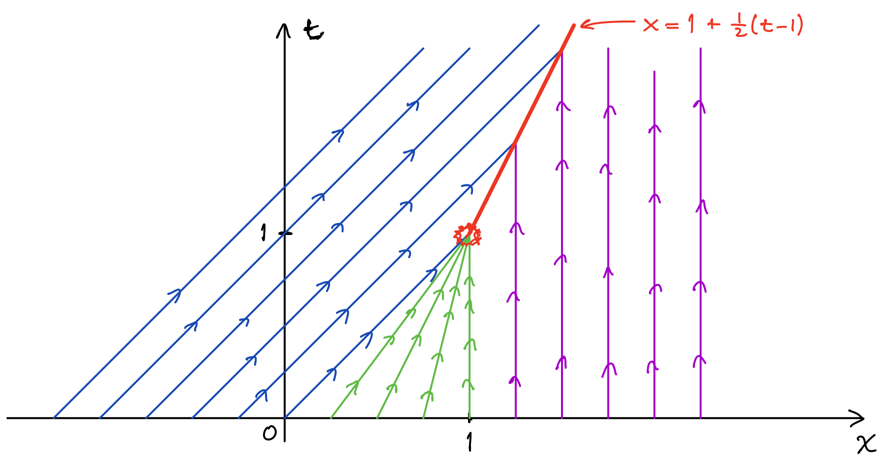
Together with the initial condition $x_s(1) = 1$, we get $x_s(t) = 1 + (t-1)/2$.

(c) Sketch the characteristics and the shock in the (x,t) plane.

Solution: A sketch is shown below.

(d) Find the solution $u(x,t)$.

Solution: The solution satisfies the implicit equation $u = u_0(x_0) = u_0(x - ut)$. When $x_0 < 0$, $u_0 = 1$, and so we have $u = 1$ along the characteristics $x_0 = x - t$ for $x_0 < 0$, provided they haven't met the shock (blue lines in diagram). Similarly, $u_0 = 0$ for $x_0 > 0$, and so $u = 0$ along the characteristics $x_0 = x$ for $x_0 > 0$ (purple lines). Finally, if $0 < x_0 < 1$ we have $u_0 = 1 - x_0$, and so $u = 1 - (x - ut)$, which yields $u = (1-x)/(1-t)$ (green lines). Putting everything together, we obtain



$$u(x,t) = \begin{cases} 1, & (t<1 \text{ and } x \le t) \text{ or } (t \ge 1 \text{ and } x > 1 + (t-1)/2), \\ (1-x)/(1-t), & t < 1 \text{ and } x > t \text{ and } x < 1, \\ 0, & (t<1 \text{ and } x \ge 1) \text{ or } (t \ge 1 \text{ and } x > 1 + (t-1)/2). \end{cases} \tag{14}$$

2. **Heat Equation.** Prove that any smooth solution, $u(x,y,t)$ in the box $\Omega = (-1,1) \times (-1,1)$ of the following equation

$$u_t = uu_x + uu_y + \Delta u, \quad t > 0, \quad (x,y) \in \Omega,$$
$$u(x,y,0) = f(x,y), \quad (x,y) \in \Omega,$$

satisfies the weak maximum principle

$$\max_{\overline{\Omega}\times[0,T]} u(x,y,t) \le \max \left\{ \max_{0 \le t \le T} u(\pm 1, \pm 1, t), \ \max_{(x,y)\in\Omega} f(x,y) \right\}.$$

Solution: Suppose u satisfies the given problem. Let $u(x,y) = v(x,y) + \epsilon t$ for $\epsilon > 0$. Then,

$$v_t + \epsilon = vv_x + vv_y + \epsilon t(v_x + v_y) + \Delta v.$$

Suppose v has a maximum at $(x_0, y_0, t_0) \in \Omega \times (0,T)$. Then,

$$v_x = v_y = v_t = 0 \quad \Rightarrow \quad \epsilon = \Delta v \quad \Rightarrow \quad \Delta v > 0,$$

at (x_0, y_0, t_0), which contradicts the maximality of v there. Thus, the maximum of v occurs on the boundary of $\Omega \times (0,T)$.

Suppose v has a maximum at (x_0, y_0, T), $(x_0, y_0) \in \Omega$. Then

$$v_x = v_y = 0, \quad v_t \ge 0 \quad \Rightarrow \quad \epsilon \le \Delta v \quad \Rightarrow \quad \Delta v > 0,$$

at (x_0, y_0, T), again a contradiction.

Combining these two results, we have that the maximum of v occurs on the boundary $\partial\Omega$ or at $t = 0$

$$\max_{\overline{\Omega}\times[0,T]} v(x,y,t) \le \max \left\{ \max_{0 \le t \le T} v(\pm 1, \pm 1, t), \ \max_{(x,y)\in\Omega} f(x,y) \right\}.$$

We can therefore compute

$$\begin{aligned} \max_{\overline{\Omega}\times[0,T]} u(x,y,t) &= \max_{\overline{\Omega}\times[0,T]} (v(x,y,t) + \epsilon t) \\ &\le \max_{\overline{\Omega}\times[0,T]} v(x,y,t) + \epsilon T \\ &\le \max \left\{ \max_{0 \le t \le T} v(\pm 1, \pm 1, t), \ \max_{(x,y)\in\Omega} f(x,y) \right\} + \epsilon T \\ &\le \max \left\{ \max_{0 \le t \le T} u(\pm 1, \pm 1, t), \ \max_{(x,y)\in\Omega} f(x,y) \right\} + \epsilon T. \end{aligned}$$

Letting $\epsilon \to 0$ establishes the result.

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3. **Wave Equation.** Consider the following initial-boundary value problem on the domain $D = \{(x,t) : t \in \mathbb{R}^+, x \in \mathbb{R}^+, x > t/\alpha\}$, where $\alpha > 1$:

$$\begin{aligned} &u_{tt} = u_{xx}, && x > t/\alpha,\ t > 0, && (15)\\ &u(x,0) = \phi(x), && x > 0, && (16)\\ &u_t(x,0) = \psi(x), && x > 0, && (17)\\ &u(x,\alpha x) = f(x), && x > 0, && (18)\end{aligned}$$

with $\phi,\ \psi,\ f \in \mathcal{C}^2(\mathbb{R}_0^+)$.

(a) Find the solution $u(x,t)$.

Solution: We seek a solution of the form

$$u(x,t) = F(x-t) + G(x+t). \qquad (19)$$

Using the initial conditions, we find

$$\begin{aligned} u(x,0) &= F(x) + G(x) = \phi(x), \quad x > 0, && (20)\\ u_t(x,0) &= -F'(x) + G'(x) = \psi(x), \quad x > 0. && (21)\end{aligned}$$

Integrating the second equation and solving, we find

$$\begin{aligned} F(x) &= \frac{1}{2}\phi(x) - \frac{1}{2}\int_0^x \psi(y)dy - \frac{1}{2}A, \quad x > 0, && (22)\\ G(x) &= \frac{1}{2}\phi(x) + \frac{1}{2}\int_0^x \psi(y)dy + \frac{1}{2}A, \quad x > 0, && (23)\end{aligned}$$

where A is the integration constant. Therefore, for $x-t > 0$ we have D'Alembert's solution

$$u(x,t) = F(x-t) + G(x+t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)dy. \qquad (24)$$

To find the solution when $t/\alpha < x < t$, we need to find $F(x)$ when $x < 0$. Using the boundary condition $u(x,\alpha x) = f(x)$ for $x > 0$, we obtain

$$u(x,\alpha x) = F((1-\alpha)x) + G((1+\alpha)x) = f(x), \qquad (25)$$

from which we get

$$F((1-\alpha)x) = f(x) - G((1+\alpha)x), \quad x > 0, \qquad (26)$$

Letting $z = (1-\alpha)x < 0$,

$$F(z) = f\left(\frac{z}{1-\alpha}\right) - G\left(\frac{1+\alpha}{1-\alpha}z\right), \quad z < 0. \qquad (27)$$

So, for $x < t$ the solution is given by

$$\begin{aligned} u(x,t) = F(x-t) + G(x+t) &= f\left(\frac{x-t}{1-\alpha}\right) - G\left(\frac{1+\alpha}{1-\alpha}(x-t)\right) + G(x+t) && (28)\\ &= f\left(\frac{x-t}{1-\alpha}\right) + \frac{1}{2}\left[\phi(x+t) - \phi\left(\frac{1+\alpha}{1-\alpha}(x-t)\right)\right] + \frac{1}{2}\int_{\frac{1+\alpha}{1-\alpha}(x-t)}^{x+t} \psi(y)dy. && (29)\end{aligned}$$

In summary,

$$u(x,t) = \begin{cases} \frac{1}{2}[\phi(x+t) + \phi(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)dy, & x > t, \\ f\left(\frac{x-t}{1-\alpha}\right) + \frac{1}{2}\left[-\phi\left(\frac{1+\alpha}{1-\alpha}(x-t)\right) + \phi(x+t)\right] + \frac{1}{2}\int_{\frac{1+\alpha}{1-\alpha}(x-t)}^{x+t} \psi(y)dy, & x < t. \end{cases} \qquad (30)$$

(b) Find sufficient conditions on ψ, ϕ, and f so that the solution is continuous in D.

Solution: We need to ensure continuity across $x = t$, where the two solutions meet. Letting $x \to t^+$ and using the fact that the functions involved are continuous we get

$$\lim_{x\to t^+} u(x,t) = \frac{1}{2}[\phi(2t) + \phi(0)] + \frac{1}{2}\int_0^{2t} \psi(y)dy. \tag{31}$$

Now taking the limit on the other side we get

$$\lim_{x\to t^-} u(x,t) = f(0) + \frac{1}{2}[\phi(2t) - \phi(0)] + \frac{1}{2}\int_0^{2t} \psi(y)dy. \tag{32}$$

Therefore, for continuity we need $\phi(0)/2 = f(0) - \phi(0)/2$, which implies $f(0) = \phi(0)$.

4. **Laplace's Equation/Green's Functions.** Consider the Neumann problem on the disk in $\mathbb{R}^2$

$$\begin{aligned} \Delta u(\mathbf{x}) = 0, \quad &\mathbf{x} \in B(0,1) = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\right\}, \\ \frac{\partial u}{\partial r}(r=1,\theta) = g(\theta), \quad &\theta \in [0,2\pi], \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi), \end{aligned} \tag{33}$$

where $r = |\mathbf{x}|$ and $\theta = \arctan(x_2/x_1)$ are polar coordinates and $g \in C^2(0,2\pi)$.

(a) What is a necessary condition for the solution to exist? What additional condition can be applied to make the solution unique? Prove that under this condition, the solution is unique.

(b) Solve the Neumann problem in (33).

(c) Using your solution from (b), identify the Neumann function for the unit disk. *Hint:* $\sum_{n=1}^{\infty} R^n/n = -\log(1-R)$ *for* $|R| < 1$.

Solution:

(a)

$$0 = \int_{B(0,1)} \Delta u(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_0^{2\pi} u_r(1,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} g(\theta)\,\mathrm{d}\theta,$$

yields the zero average condition on the Neumann data g. This is a necessary condition for existence.

For uniqueness, the Neumann problem is determined up to an overall constant. We can fix this constant in many ways. One way is to require

$$\int_0^{2\pi} u(1,\theta)\,\mathrm{d}\theta = 0,$$

or, equivalently by the mean value theorem, $u(\mathbf{0}) = 0$. An alternative is to fix the value of u at a specific point in the domain $u(\mathbf{x}_0) = u_0 \in \mathbb{R}$ for some $\mathbf{x}_0 \in \overline{B(0,1)}$.

To prove uniqueness, we suppose two solutions solving the same problem $u_j(\mathbf{x})$, $j = 1, 2$. Then $v(\mathbf{x}) = u_2(\mathbf{x}) - u_1(\mathbf{x})$ solves the homogeneous problem

$$\begin{aligned} \Delta v(\mathbf{x}) = 0, \quad &\mathbf{x} \in B(0,1) = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\right\}, \\ \frac{\partial v}{\partial r}(r=1,\theta) = 0, \quad &\theta \in [0,2\pi], \quad \int_0^{2\pi} v(1,\theta)\,\mathrm{d}\theta = 0. \end{aligned}$$

Multiplying the PDE by v and integrating over the domain, we have

$$\begin{aligned} 0 &= \int_{B(0,1)} v(\mathbf{x})\Delta v(\mathbf{x})\,\mathrm{d}\mathbf{x} \\ &= -\int_{B(0,1)} |\nabla v(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} + \int_0^{2\pi} v(1,\theta)v_r(1,\theta)\,\mathrm{d}\theta \\ &= -\int_{B(0,1)} |\nabla v(\mathbf{x})|^2\,\mathrm{d}\mathbf{x}, \end{aligned}$$

upon applying integration by parts and the boundary condition. Since the integrand is non-negative definite and the integral is zero, we must have

$$|\nabla v(\mathbf{x})|^2 = 0, \quad \mathbf{x} \in B(0,1) \quad \Rightarrow \quad v(\mathbf{x}) = const, \quad \mathbf{x} \in B(0,1).$$

Since the average of $v(\mathbf{x})$ on the boundary is zero, $v(\mathbf{x})$ must be identically zero and uniqueness is proven.

(b) We seek a solution using the method of separation of variables in polar coordinates. Then, eq. (33) becomes

$$\begin{aligned} & u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad r \in (0,1), \quad \theta \in (0,2\pi), \\ & u_r(1,\theta) = g(\theta), \quad \theta \in [0,2\pi]. \end{aligned}$$

Seeking a solution in separated form $u(r,\theta) = f(r)g(\theta)$ implies

$$\begin{aligned} & g''(\theta) + \lambda g(\theta) = 0, \quad \theta \in (0,2\pi), \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi), \\ & f''(r) + \frac{1}{r}f'(r) - \frac{\lambda}{r^2}f(r) = 0, \quad r \in (0,1), \quad \lim_{r\to 0}|f(r)| < \infty. \end{aligned}$$

The angular boundary value problem has the trigonometric solutions

$$g_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta), \quad n = 0,1,2,\ldots.$$

with the corresponding eigenvalues $\lambda_n = n^2$.
The radial problem exhibits the bounded solutions

$$f_n(r) = r^n.$$

Introduce the series solution

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)].$$

The coefficients are determined by the boundary conditions

$$u_r(1,\theta) = \sum_{n=1}^{\infty} n[A_n\cos(n\theta) + B_n\sin(n\theta)] = g(\theta), \quad \theta \in [0,2\pi].$$

Multiplying by $\cos(m\theta)$ and integrating from 0 to 2π, we obtain

$$A_m = \frac{1}{m\pi}\int_0^{2\pi} g(\theta)\cos(m\theta)\,\mathrm{d}\theta, \quad m = 1,2,\ldots.$$

Multiplying by $\sin(m\theta)$ and integrating from 0 to 2π, we obtain

$$B_m = \frac{1}{m\pi}\int_0^{2\pi} g(\theta)\sin(m\theta)\,\mathrm{d}\theta, \quad m = 1,2,\ldots,$$

which determines a series representation of the solution. To determine A_0, we require zero average on the boundary so that $A_0 = 0$.

(c) Inserting the expressions for the coefficients into the series representation, we obtain

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} \frac{r^n}{n}\frac{1}{\pi}\int_0^{2\pi} g(\phi)\Big[\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta)\Big]\,\mathrm{d}\phi \\ &= \int_0^{2\pi} g(\phi)\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n}\cos(n(\phi-\theta))\,\mathrm{d}\phi \\ &= \int_0^{2\pi} g(\phi)N(r,\theta-\phi)\,\mathrm{d}\phi. \end{aligned}$$

The series can be summed using the hint

$$\begin{aligned} N(r,\theta) &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n}\cos(n\theta) \\ &= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{r^n}{n}\frac{e^{in\theta}+e^{-in\theta}}{2} \\ &= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\frac{\left(re^{i\theta}\right)^n}{n}+\frac{\left(re^{-i\theta}\right)^n}{n}\right) \\ &= -\frac{1}{2\pi}\Big(\log(1-re^{i\theta})+\log(1-re^{-i\theta})\Big) \\ &= -\frac{1}{2\pi}\log(1-2r\cos\theta+r^2). \end{aligned}$$

Then $N(r,\theta) = -\frac{1}{2\pi}\log(1-2r\cos\theta+r^2)$ is the Neumann function for the unit disk.

5. **Solution methods.** Let $\Omega = (0,1)\times\mathbb{R}^+$, and assume that $u(x,t)\in\mathcal{C}^1(\bar{\Omega})\cap\mathcal{C}^2(\Omega)$ satisfies

$$u_t = u_{xx} + f(x)e^{-t}, \qquad 0<x<1,\ t>0, \tag{34}$$
$$u(x,0) = 0, \qquad 0<x<1, \tag{35}$$
$$u(0,t) = u(1,t) = 0 \qquad t>0, \tag{36}$$

where $f\in\mathcal{C}^1([0,1])$.

(a) Use Duhamel's principle to find a formal solution to the initial boundary value problem in terms of f_n, the Fourier coefficients of $f(x)$.

Solution: Applying Duhamel's principle, we set up the family of initial boundary value problems

$$\tilde{u}_t = \tilde{u}_{xx}, \qquad 0<x<1,\ t>s, \tag{37}$$
$$\tilde{u}(x,s;s) = f(x)e^{-s}, \qquad 0<x<1, \tag{38}$$
$$\tilde{u}(0,t;s) = u(1,t;s) = 0, \qquad t>s, \tag{39}$$

and recover the solution from

$$u(x,t) = \int_0^t \tilde{u}(x,t;s)ds. \tag{40}$$

For each problem, we use separation of variables, i.e., we set $\tilde{u}(x,t;s) = X(x)T(t)$, obtaining

$$\frac{\dot{T}}{T} = \frac{X''}{X} = a, \tag{41}$$

where a is a constant and the dot and prime indicate time and space derivatives, respectively. If $a = 0$, the spatial equation gives $X = A + Bx$, which upon evaluation of the boundary conditions leads to $X = 0$. Similarly, if $a > 0$ we get $X = Ae^{\sqrt{a}x} + Be^{-\sqrt{a}x}$, leading also to $X = 0$. Therefore, a must be negative and we set $a = -\lambda^2$. We obtain

$$T(t) = T(0)\exp(-\lambda^2 t), \tag{42}$$
$$X(x) = A\sin(\lambda x) + B\cos(\lambda x). \tag{43}$$

Using the boundary conditions $X(0) = X(1) = 0$ we obtain $B = 0$ and $\lambda = n\pi$, so we get the modes

$$X_n(x) = \sin(\lambda_n x), \tag{44}$$

where $\lambda_n = n\pi$ and $n \in \mathbb{N}^+$. Thus, we find

$$\tilde{u}(x,t;s) = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 t}\sin(\lambda_n x). \tag{45}$$

Using the initial conditions $\tilde{u}(x,t;s) = f(x)e^{-s}$ we get

$$f(x)e^{-s} = \sum_{n=1}^{\infty} A_n e^{-\lambda_n^2 s}\sin(\lambda_n x), \tag{46}$$

which implies that $A_n = f_n e^{(\lambda_n^2 - 1)s}$, where f_n is the nth sine Fourier coefficient of $f(x)$. Therefore,

$$\tilde{u}(x,t;s) = \sum_{n=1}^{\infty} f_n e^{(\lambda_n^2-1)s} e^{-\lambda_n^2 t}\sin(\lambda_n x). \tag{47}$$

and

$$u(x,t) = \int_0^t \tilde{u}(x,t;s)ds = \int_0^t \sum_{n=1}^{\infty} f_n e^{(\lambda_n^2-1)s} e^{-\lambda_n^2 t}\sin(\lambda_n x)ds \tag{48}$$
$$= \sum_{n=1}^{\infty} f_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\int_0^t e^{(\lambda_n^2-1)s}ds \tag{49}$$
$$= \sum_{n=1}^{\infty} f_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\frac{e^{(\lambda_n^2-1)s}}{\lambda_n^2 - 1}\Big|_0^t \tag{50}$$
$$= \sum_{n=1}^{\infty} f_n e^{-\lambda_n^2 t}\sin(\lambda_n x)\frac{e^{(\lambda_n^2-1)t} - 1}{\lambda_n^2 - 1} \tag{51}$$
$$= \sum_{n=1}^{\infty} f_n \sin(\lambda_n x)\frac{e^{-t} - e^{-\lambda_n^2 t}}{\lambda_n^2 - 1}. \tag{52}$$

(b) Prove that the solution is unique.

Solution: Assume there are two solutions, u_1 and u_2. Then their difference $w = u_1 - u_2$ satisfies

$$w_t = w_{xx}, \qquad 0 < x < 1,\ t > 0, \tag{53}$$
$$w(x,0) = 0, \qquad 0 < x < 1, \tag{54}$$
$$w(0,t) = u(1,t) = 0 \qquad t > 0. \tag{55}$$

Let $T > 0$. By the maximum principle, the maximum of w in the closure of $U_T = [0,1] \times [0,T)$ must be equal to the maximum of w in its parabolic boundary, $\bar{U}_T - U_T$, which is zero. Therefore $w \leq 0$, or equivalently $u_1 \leq u_2$ in $\bar{U}_T$. Applying the same argument to $-w$ we conclude that $w = u_1 - u_2 \equiv 0$ in $\bar{U}_T$. Since T was arbitrary, $u_1(x,t) = u_2(x,t)$ for all $t > 0$, $x \in (0,1)$, so the solution is unique.

♢	♠	**END**	♡	♣