Preliminary Exam
Partial Differential Equations
9:00 AM - 12:00 PM, Jan. 11, 2024
Newton Lab, ECCR 257

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.** Each problem is worth 25 points.

A sheet of convenient formulae is provided.

1. **Method of characteristics.** Consider the inviscid Burger's equation

$$u_t + uu_x = 0 \tag{1}$$

on the domain $\Omega = \mathbb{R} \times \mathbb{R}^+$ with initial conditions

$$u(x,0) = u_0(x) = \begin{cases} 1, & x \le 0, \\ 1-x, & 0 < x \le 1, \\ 0, & 1 < x. \end{cases} \tag{2}$$

   (a) Find the time and position at which a shock forms.

   (b) Find the subsequent trajectory of the discontinuous shock by applying the Rankine-Hugoniot condition

$$s(t) = \frac{1}{2}(u_-(t) + u_+(t)),$$

   where s is the speed of the discontinuity and $u_\pm(t) = \lim_{x \to x_s(t)^\pm} u(x,t)$ and $s = \dot{x}_s(t)$.

   (c) Sketch the characteristics and the shock in the (x,t) plane.

   (d) Find the solution $u(x,t)$.

2. **Heat Equation.** Prove that any smooth solution, $u(x,y,t)$ in the box $\Omega = (-1,1) \times (-1,1)$ of the following equation

$$u_t = uu_x + uu_y + \Delta u, \quad t > 0, \quad (x,y) \in \Omega,$$
$$u(x,y,0) = f(x,y), \quad (x,y) \in \Omega,$$

satisfies the weak maximum principle

$$\max_{\overline{\Omega}\times[0,T]} u(x,y,t) \le \max\left\{ \max_{0 \le t \le T} u(\pm 1, \pm 1, t), \ \max_{(x,y)\in\Omega} f(x,y) \right\}.$$

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3. **Wave Equation.** Consider the following initial-boundary value problem on the domain $D = \{(x,t) : t \in \mathbb{R}^+, x \in \mathbb{R}^+, x > t/\alpha\}$, where $\alpha > 1$:

$$\begin{aligned} u_{tt} &= u_{xx}, && x > t/\alpha,\ t > 0, && (3)\\ u(x,0) &= \phi(x), && x > 0, && (4)\\ u_t(x,0) &= \psi(x), && x > 0, && (5)\\ u(x,\alpha x) &= f(x), && x > 0, && (6)\end{aligned}$$

with $\phi,\ \psi,\ f \in \mathcal{C}^2(\mathbb{R}_0^+)$.

(a) Find the solution $u(x,t)$.

(b) Find sufficient conditions on ψ, ϕ, and f so that the solution is continuous in D.

4. **Laplace's Equation/Green's Functions.** Consider the Neumann problem on the disk in $\mathbb{R}^2$

$$\begin{aligned} \Delta u(\mathbf{x}) = 0, \quad \mathbf{x} \in B(0,1) = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 1\right\}, \\ \frac{\partial u}{\partial r}(r=1,\theta) = g(\theta), \quad \theta \in [0, 2\pi], \quad g(0) = g(2\pi), \quad g'(0) = g'(2\pi), \end{aligned} \tag{7}$$

where $r = |\mathbf{x}|$ and $\theta = \arctan(x_2/x_1)$ are polar coordinates and $g \in C^2(0, 2\pi)$.

(a) What is a necessary condition for the solution to exist? What additional condition can be applied to make the solution unique? Prove that under this condition, the solution is unique.

(b) Solve the Neumann problem in (7).

(c) Using your solution from (b), identify the Neumann function for the unit disk. *Hint:* $\sum_{n=1}^{\infty} R^n/n = -\log(1-R)$ *for* $|R| < 1$.

5. **Solution methods.** Let $\Omega = (0,1) \times \mathbb{R}^+$, and assume that $u(x,t) \in \mathcal{C}^1(\bar{\Omega}) \cap \mathcal{C}^2(\Omega)$ satisfies

$$\begin{aligned} u_t &= u_{xx} + f(x)e^{-t}, && 0 < x < 1,\ t > 0, && (8)\\ u(x,0) &= 0, && 0 < x < 1, && (9)\\ u(0,t) &= u(1,t) = 0 && t > 0, && (10)\end{aligned}$$

where $f \in \mathcal{C}^1([0,1])$.

(a) Use Duhamel's principle to find a formal solution to the initial boundary value problem in terms of f_n, the Fourier coefficients of $f(x)$.

(b) Prove that the solution is unique.

♢	♠	**END**	♡	♣