1. **Quasilinear first order equations.** The density of cars $\rho(x, t)$ in a traffic model satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} [\rho(1 - \rho)] = 0, \quad t > 0, \quad -\infty < x < \infty$$

with initial conditions

$$\rho(x, 0) = \rho_0(x) = \begin{cases} 1 - x^2, & -1 < x < 1, \\ 0, & |x| \geq 1. \end{cases}$$

(a) (15 points) Find $\rho(x, t)$ for times $t$ less than the time at which a shock forms.

(b) (10 points) Find the time at which a shock forms.

**Solution:**

(a) The PDE can be rewritten as

$$\frac{\partial \rho}{\partial t} + (1 - 2\rho) \frac{\partial \rho}{\partial x} = 0. \quad (1)$$

The characteristics are determined by the equations

$$\frac{dt}{d\tau} = 1, \quad (2)$$
$$\frac{dx}{d\tau} = 1 - 2\rho, \quad (3)$$
$$\frac{d\rho}{d\tau} = 0. \quad (4)$$

Inserting the initial conditions $t(0) = 0$, $\rho(0) = \rho_0(x_0)$, $x(0) = x_0$, we find that $\rho$ is constant along the characteristics

$$x = x_0 + [1 - 2\rho_0(x_0)]t.$$ 

Therefore the density $\rho$ at $(x, t)$, provided no shock has formed, satisfies

$$\rho = \rho_0(x_0) = \rho_0(x - [1 - 2\rho_0(x_0)]t).$$
For \((x, t)\) such that \(x_0 < -1\) or \(x_0 > 1\), \(\rho = 0\). These correspond to \(x < -1 + t\) and \(x > 1 + t\), respectively. For \((x, t)\) such that \(|x_0| < 1\), we have

\[
\rho = \rho_0(x_0) = 1 - (x - [1 - 2\rho]t)^2.
\]

Solving the quadratic, we get

\[
\rho(x, t) = \frac{4t^2 - 4tx - 1 \pm \sqrt{8t^2 + 8tx + 1}}{8t^2}.
\]

Choosing the positive sign so that \(\lim_{(x, t) \to (0, 0)} \rho(x, t) = \rho_0(0) = 1\), we obtain that provided no shock has formed,

\[
\rho(x, t) = \begin{cases} 
0, & x < -1 + t \text{ or } x > 1 + t \\
\frac{4t^2 - 4tx - 1 + \sqrt{8t^2 + 8tx + 1}}{8t^2}, & \text{otherwise}.
\end{cases}
\]  

(5)

(b) When \(x_0 < -1\) the slope of the characteristics is \(+1\) and when \(x = 1\) the slope is \(-1\), so characteristics will cross at some point and there will be a shock (see Figure). To find the time at which the shock forms, we can proceed by taking a partial \(x\) derivative of (1):

\[
\rho_xt - 2\rho^2_x + (1 - 2\rho)\rho_{xx} = 0.
\]

Along characteristics, \(\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t} \frac{\partial}{\partial x} = \frac{\partial}{\partial t} + [1 - 2\rho_0(x_0)] \frac{\partial}{\partial x}\), so \(\rho_x + (1 - 2\rho)\rho_{xx} = d\rho_x/dt\).

Letting \(y = \rho_x\), we get

\[
\frac{dy}{dt} = 2y^2.
\]

Solving this ODE with the initial conditions \(y(0) = \rho_x(x_0)\) we get

\[
y(t) = \begin{cases} 
\frac{-2x_0}{1 + 4tx_0}, & -1 < x < 1, \\
0, & |x| \geq 1.
\end{cases}
\]

(6)

The time at which \(y(t)\) diverges is \(t = -1/(4x_0)\), which is minimized at \(x_0 = -1\). Therefore the shock forms at \(t = 1/4\). (You can also directly use the formula \(t = -1/(2\min(\rho'_0))\).)

Figure 1: Characteristics given by \(x = x_0 + [1 - 2\rho_0(x_0)]t\)
2. **Heat Equation.** Consider the heat equation in an infinite rod

\[ w_t = w_{xx}, \quad -\infty < x < \infty, t > 0 \]

\[ w(x,0) = f(x), \]

where \( f(x) \in C(\mathbb{R}) \) is zero for \(|x| > L\).

(a) (13 points) Show that \( \int_{-\infty}^{\infty} w \, dx \) is independent of time.

(b) (12 points) Show that \( q_n(t) = \int_{-\infty}^{\infty} x^{2n} w \, dx \) is a polynomial of degree \( n \) in \( t \) for \( n \geq 0 \).

**Solution:**

(a) We first show that \( q_0 = \int_{-\infty}^{\infty} w \, dx \) is independent of time. Taking a time derivative, we get

\[
\frac{dq_0}{dt} = \int_{-\infty}^{\infty} w_t \, dx = \int_{-\infty}^{\infty} w_{xx} \, dx = w_x \bigg|_{-\infty}^{\infty}.
\]

The solution \( w \) and its spatial derivative \( w_x \) can be expressed in terms of the fundamental solution as

\[
w = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y) \, dy,
\]

\[
w_x = \int_{-\infty}^{\infty} -\frac{2(x-y)}{4t\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y) \, dy,
\]

and since \( f(y) = 0 \) for \(|y| > L\), we have that for \( x > L \) and larger than the value at which \((x-L)\exp(-(x-L)^2/4t)\) is maximized

\[
|w_x| \leq \int_{-\infty}^{\infty} \frac{2|x-y|}{4t} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} |f(y)| \, dy < \frac{2(x+L)}{4t} \frac{1}{\sqrt{4\pi t}} e^{-(x-L)^2/4t} \int_{-L}^{L} |f(y)| \, dy.
\]

Since \( f \in C([-L,L]) \) and the function multiplying the integral goes to zero as \( x \to \infty \) for any fixed \( t \), \( \lim_{x \to \infty} |w_x| = 0 \). Similarly, \( \lim_{x \to -\infty} |w_x| = 0 \), so

\[
\frac{dq_0}{dt} = 0,
\]

and \( q_0(t) \) is a constant, i.e., a polynomial of degree 0.

(b) We proceed by induction. The case \( n = 0 \) was shown in (a). For the inductive step, assume \( q_n(t) \) is a polynomial of degree \( n \) in \( t \). Now let’s calculate \( \frac{dq_{n+1}}{dt} \):

\[
\frac{dq_{n+1}}{dt} = \int_{-\infty}^{\infty} x^{2n+2} w_t \, dx = \int_{-\infty}^{\infty} x^{2n+2} w_{xx} \, dx.
\]

Integrating by parts twice,

\[
\frac{q_{n+1}}{dt} = x^{2n+2} w_x \bigg|_{-\infty}^{\infty} - (2n+2)x^{2n+1} w \bigg|_{-\infty}^{\infty} + (2n+1)(2n+2) \int_{-\infty}^{\infty} x^{2n} w \, dx.
\]
Like before, one can show that \(x^{2n+2}w_x\bigg|_{-\infty}^{\infty} - (2n + 2)x^{2n+1}w\bigg|_{-\infty}^{\infty} = 0\), and so
\[
\frac{q_{n+1}}{dt} = (2n + 1)(2n + 2)q_n,
\]
which using the induction hypothesis is a polynomial of degree \(n\) in \(t\). Integrating with respect to \(t\), we find that \(q_{n+1}\) is a polynomial of degree \(n + 1\) in \(t\).
3. Wave Equation. Consider the following initial boundary value problem

\[ u_{tt} = c^2 u_{xx}, \quad x > 0, \quad t > 0, \]
\[ u(x, 0) = 0, \quad u_t(x, 0) = g(x), \]
\[ u_x(0, t) = 0. \]

(a) (10 points) Find all solutions to the above IBVP that lie in \( x > 0, \ t > 0. \)

\textbf{Solution:} By d’Alembert’s approach, we know for \( x > ct, \)
\[ u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(y) dy, \]
but for \( x < ct, \) consider the interaction of the initial condition with the boundary. To do so, note the solutions take the form
\[ u(x, t) \equiv F(x + ct) + G(x - ct), \]
so \( F(x) + G(x) = 0 \) and \( cF'(x) - cG'(x) = g(x) \) and \( F'(ct) + G'(-ct) = F'(x) + G'(-x) = 0. \) Integrating the 2nd and 3rd expressions yields

\[ F(x) - G(x) = \frac{1}{c} \int_0^x g(y) dy + F(0) - G(0), \]
\[ F(x) - G(-x) = F(0) - G(0). \]

Applying the 1st expression \((G(x) = -F(x))\) to integrated 2nd expression yields
\[ F(x) = \frac{1}{2c} \int_0^x g(y) dy + F(0). \]
Rearranging the last expression then implies
\[ G(x) = F(-x) - 2F(0) = \frac{1}{2c} \int_{-x}^0 g(y) dy - F(0). \]
Thus, for \( x < ct, \)
\[ u(x, t) = F(x + ct) + G(x - ct) = \frac{1}{2c} \left[ \int_0^{x+ct} g(y) dy + \int_0^{ct-x} g(y) dy \right]. \]

(b) (10 points) Using an energy functional, show if a solution to the IBVP exists, it is unique, given adequate assumptions. State these needed assumptions.

\textbf{Solution:} Define the following energy functional \( E(t) \equiv \frac{1}{2} \int_0^\infty u_x^2(x, t) + c^2 u_x^2(x, t) dx \)
and assume there are two solutions to the IBVP, \( u \) and \( v \) and take \( w \equiv u - v. \) First, note by linearity, \( w \) solves the same IBVP:

\[ w_{tt} = c^2 w_{xx}, \quad w(x, 0) \equiv w_i(x, 0) \equiv 0, \quad w_x(0, t) \equiv 0. \]
Thus, passing \( w(x, t) \) through the energy functional and differentiating with respect to time, we obtain

\[ E'(t) \equiv \frac{d}{dt} \left[ \int_0^\infty u_x^2(x, t) + c^2 u_x^2(x, t) dx \right] = \int_0^\infty w_t(x, t) w_{tt}(x, t) dx + c^2 \int_0^\infty w_x(x, t) w_{xt}(x, t) dx \]
\[ = \int_0^\infty w_t(x, t)[w_{tt}(x, t) - c^2 w_{xx}(x, t)] dx = 0, \]
Poisson’s Equation/Green’s Functions.

(4.1) Consider Poisson’s equation with homogeneous Dirichlet boundary conditions in the half 2D ball:

$$\Delta u(x) = f(x), \quad x \in \Omega = \{ x \in \mathbb{R}^2 \mid x_2 > 0 & ||x|| < 1 \},$$

$$u(x) = g(x), \quad x \in \partial \Omega.$$  

Determine the associated Green’s function $$G(x, y)$$ in terms of the fundamental solution $$\Phi(||x||) = \frac{1}{2\pi} \log ||x||$$, and write the solution to the above boundary value problem.

**Solution:** Define $$\tilde{x} = (x_1, -x_2), \tilde{x} = x/||x||^2$$, and $$x^* = \tilde{x}/||x||^2$$ then

$$G(x, y) = \Phi(||x - y||) - \Phi(||\tilde{x} - y||) - \Phi(||x|| \cdot ||\tilde{x} - y||) + \Phi(||x|| \cdot ||x^* - y||).$$

The solution to the boundary value problem is then

$$u(x) = \int_{\Omega} G(x, y)f(y)dy - \int_{\partial \Omega} \frac{\partial G}{\partial n}(x, y)g(y)dS_y.$$  

(4.2) Use the maximum principle to prove the uniqueness of the solution in part (a).

**Solution:** Assume $$u$$ and $$v$$ both solve the BVP, then $$w = u - v$$ solves

$$-\Delta w(x) = 0, \quad x \in \Omega$$

$$w(x) = 0, \quad x \in \partial \Omega.$$  

By the maximum principle, since $$\Omega$$ is simply connected, any smooth solution to the BVP has its maximum and minimum on $$\partial \Omega$$ which is zero throughout, so $$w \equiv 0$$, implying $$u \equiv v$$.

(4.3) Assume $$f \equiv 0$$ in the above BVP and prove for any ball $$B_r(x)$$ of radius $$r$$ in $$\Omega$$:

$$u(x) = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)dS_y$$  

for all $$B_r(x) \subset \Omega$$.

**Solution:** Let

$$h(r) := \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} u(y)dS_y = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(x + rz)dS_z$$

Then

$$h'(r) = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} z \cdot \nabla u(x + rz)dS_z = \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} \frac{y - x}{r} \cdot \nabla u(y)dS_y$$

$$= \frac{1}{|\partial B_r(x)|} \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu}dS_y = \frac{1}{|\partial B_r(x)|} \int_{B_r(x)} \Delta u(y)dS_y = 0,$$

where $$\nu$$ is the unit normal and by the divergence theorem. This implies the given integral is constant for all $$r$$. In the limit as $$r \to 0$$, note $$h(r) \to u(x)$$ for a smooth function $$u$$. 


5. **Separation of Variables.** Consider Laplace’s equation in the sector $W = \{(r, \theta) : 1 < r < a, 0 < \theta < \alpha\} \subseteq \mathbb{R}^2$

\[ \Delta u = 0, \quad x \in W, \]

with boundary conditions

\[ u(r, 0) = u(r, \theta_0) = 0, \]
\[ u(1, \theta) = 0, \]
\[ u(a, \theta) = f(\theta). \]

(a) (17 points) Find a formal solution $u(r, t)$ of the above boundary value problem in terms of $f$.

(b) (8 points) Find nontrivial conditions on $g$ that guarantee that the solution you found is a classical solution $u$ in $C^2(\bar{W})$.

**Solution:**

(a) In polar coordinates, we have

\[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0. \]

Using separation of variables we set $u(r, \theta) = R(r) \Theta(\theta)$ and obtain

\[ R'' + \frac{1}{r} R' + \frac{1}{r^2} R \dot{\Theta} = 0, \]

where prime and dot are derivatives with respect to $r$ and $\theta$, respectively. Separating variables we get

\[ r^2 R'' + r R' = aR, \]
\[ \ddot{\Theta} = -a \Theta. \]  

First consider the angular equation. If $a < 0$ we get $\Theta(\theta) = A e^{\sqrt{-a} \theta} + B e^{-\sqrt{-a} \theta}$, and from the boundary conditions $\Theta(0) = \Theta(\alpha) = 0$ we get $A = B = 0$. If $a = 0$ we get $\Theta(\theta) = A + B \theta$ and again we obtain $A = B = 0$. To avoid a trivial solution, $a$ must be positive, and we get $\Theta(\theta) = A \sin(\sqrt{a} \theta) + B \cos(\sqrt{a} \theta)$. The boundary condition $\Theta(0) = 0$ gives $B = 0$, and $\Theta(\alpha) = 0$ gives $\sqrt{a} = 2n\pi/\alpha$, with $n \in \mathbb{Z}$. So we obtain the solutions

\[ \sin \left( \frac{2\pi n \theta}{\alpha} \right), \quad n \in \mathbb{Z}. \]  

Now consider the radial equation. We have $r^2 R'' + r R' - aR = 0$. Inserting the ansatz $R = r^\mu$ we get $\mu(\mu - 1) + \mu - a = 0$, and so $\mu = \pm \sqrt{a} = \pm 2n\pi/\alpha$. Therefore we obtain the solutions

\[ C_n r^{2n\pi/\alpha} + D_n r^{-2n\pi/\alpha}. \]

Now we construct the general solution as

\[ u(r, \theta) = \sum_{n=1}^{\infty} \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ C_n r^{2n\pi/\alpha} + D_n r^{-2n\pi/\alpha} \right]. \]
(Note that $n = 0$ does not contribute and $n < 0$ are redundant).

Now we need to choose $C_n$ and $D_n$ to satisfy the boundary conditions at $r = 1$ and $r = a$. At $r = 1$

$$u(1, \theta) = 0 = \sum_{n=1}^{\infty} \sin \left( \frac{2\pi n \theta}{\alpha} \right) [C_n + D_n].$$

Since this is true for all $\theta$ in $[0, \alpha]$, by orthogonality of $\{\sin \left( \frac{2\pi n \theta}{\alpha} \right)\}_{n \in \mathbb{N}}$ in $[0, \alpha]$ we must have

$$C_n + D_n = 0.$$

Therefore the solution is

$$u(r, \theta) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ r^{2n\pi/\alpha} - r^{-2n\pi/\alpha} \right].$$

Finally, we use the boundary condition at $r = a$ to find the coefficients $C_n$:

$$u(a, \theta) = f(\theta) = \sum_{n=1}^{\infty} C_n \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ a^{2n\pi/\alpha} - a^{-2n\pi/\alpha} \right].$$

Multiplying by $\sin \left( \frac{2\pi k \theta}{\alpha} \right)$ and integrating from 0 to $\alpha$ we get

$$C_k = [a^{2k\pi/\alpha} - a^{-2k\pi/\alpha}]^{-1} \frac{2}{\alpha} \int_{0}^{\alpha} f(\theta) \sin \left( \frac{2\pi k \theta}{\alpha} \right) d\theta. \quad (15)$$

(b) It is enough that $f(0) = f(\alpha)$, $f''(0) = f''(\alpha)$, $f \in C^4(\bar{W})$ for the solution to be a classical solution. To show this, we need to show that the solution (12) with coefficients (15) can be differentiated termwise as needed. We have

$$u_{\theta \theta} = -\sum_{n=1}^{\infty} C_n \left( \frac{2\pi n}{\alpha} \right)^2 \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ r^{2n\pi/\alpha} - r^{-2n\pi/\alpha} \right],$$

$$u_r = \sum_{n=1}^{\infty} C_n \left( \frac{2\pi n}{\alpha} \right) \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ r^{2n\pi/\alpha - 1} + r^{-2n\pi/\alpha - 1} \right],$$

$$u_{rr} = \sum_{n=1}^{\infty} C_n \left( \frac{2\pi n}{\alpha} \right) \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ \left( \frac{2\pi n}{\alpha} - 1 \right) r^{2n\pi/\alpha - 2} - \left( \frac{2\pi n}{\alpha} + 1 \right) r^{-2n\pi/\alpha - 2} \right].$$

If these series converge absolutely and uniformly, we can differentiate termwise. To show this, we use Weierstrass’ M-test, and note that for $1 < r < a$

$$\sum_{n=1}^{\infty} \left| C_n \left( \frac{2\pi n}{\alpha} \right)^2 \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ r^{2n\pi/\alpha} - r^{-2n\pi/\alpha} \right] \right| \leq \sum_{n=1}^{\infty} \left| C_n \right| \left( \frac{2\pi n}{\alpha} \right)^2 \left[ a^{2n\pi/\alpha} + 1 \right],$$

$$\sum_{n=1}^{\infty} \left| C_n \left( \frac{2\pi n}{\alpha} \right) \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ r^{2n\pi/\alpha - 1} + r^{-2n\pi/\alpha - 1} \right] \right| \leq \sum_{n=1}^{\infty} \left| C_n \right| \left( \frac{2\pi n}{\alpha} \right) \left[ a^{2n\pi/\alpha - 1} + 1 \right],$$

$$\sum_{n=1}^{\infty} \left| C_n \left( \frac{2\pi n}{\alpha} \right) \sin \left( \frac{2\pi n \theta}{\alpha} \right) \left[ \left( \frac{2\pi n}{\alpha} - 1 \right) r^{2n\pi/\alpha - 2} - \left( \frac{2\pi n}{\alpha} + 1 \right) r^{-2n\pi/\alpha - 2} \right] \right| \leq \sum_{n=1}^{\infty} \left| C_n \right| \left( \frac{2\pi n}{\alpha} \right) \left( \frac{2\pi n}{\alpha} + 1 \right) \left[ a^{2n\pi/\alpha - 2} + 1 \right].$$
Therefore, it is enough to show that $|C_n|(a^{2n\pi/\alpha-2} + 1) \leq \frac{M}{n^4}$ for some constant $M$. Integrating (15) by parts twice and using $f(0) = f(\alpha)$ we get

$$C_k \left[ a^{2k\pi/\alpha} - a^{-2k\pi/\alpha} \right] = -\frac{2}{\alpha} \left( \frac{\alpha}{2\pi k} \right)^2 \int_0^\alpha f''(\theta) \sin \left( \frac{2\pi k \theta}{\alpha} \right) d\theta. \quad (17)$$

Integrating by parts twice again and using $f''(0) = f''(\alpha)$ we get

$$C_k \left[ a^{2k\pi/\alpha} - a^{-2k\pi/\alpha} \right] = 2 \left( \frac{\alpha}{2\pi k} \right)^4 \int_0^\alpha f''''(\theta) \sin \left( \frac{2\pi k \theta}{\alpha} \right) d\theta. \quad (18)$$

Using $f \in C^4(\bar{W})$, we find, letting $E = \max_{[0,\alpha]} |f'''|$,

$$|C_k| \leq \left[ a^{2k\pi/\alpha} - a^{-2k\pi/\alpha} \right]^{-1} \frac{2}{\alpha} \left( \frac{\alpha}{2\pi k} \right)^4 \alpha E, \quad (19)$$

and so

$$|C_n|(a^{2n\pi/\alpha-2} + 1) \leq \frac{a^{2n\pi/\alpha-2} + 1}{a^{2n\pi/\alpha} - a^{-2n\pi/\alpha}} \frac{2}{\alpha} \left( \frac{\alpha}{2\pi n} \right)^4 \alpha E \leq \frac{M}{n^4}, \quad (20)$$

and the formal solution can be differentiated termwise as required.