Preliminary Exam
Partial Differential Equations
10:00 AM - 1:00 PM, Friday, Aug. 28, 2020
Room: ECCR 244

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.** The density of cars $\rho(x,t)$ in a traffic model satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho(1-\rho)\right] = 0, \qquad t>0, -\infty < x < \infty$$

with initial conditions

$$\rho(x,0) = \rho_0(x) = \begin{cases} 1-x^2, & -1<x<1, \\ 0, & |x| \geq 1. \end{cases}$$

(a) (15 points) Find $\rho(x,t)$ for times t less than the time at which a shock forms.

(b) (10 points) Find the time at which a shock forms.

Solution:

(a) The PDE can be rewritten as

$$\frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0. \tag{1}$$

The characteristics are determined by the equations

$$\frac{dt}{d\tau} = 1, \tag{2}$$

$$\frac{dx}{d\tau} = 1-2\rho, \tag{3}$$

$$\frac{d\rho}{d\tau} = 0. \tag{4}$$

Inserting the initial conditions $t(0)=0$, $\rho(0)=\rho_0(x_0)$, $x(0)=x_0$, we find that ρ is constant along the characteristics

$$x = x_0 + [1-2\rho_0(x_0)]t.$$

Therefore the density ρ at (x,t), provided no shock has formed, satisfies

$$\rho = \rho_0(x_0) = \rho_0(x-[1-2\rho_0(x_0)]t).$$

For (x,t) such that $x_0 < -1$ or $x_0 > 1$, $\rho = 0$. These correspond to $x < -1 + t$ and $x > 1 + t$, respectively. For (x,t) such that $|x_0| < 1$, we have

$$\rho = \rho_0(x_0) = 1 - (x - [1 - 2\rho]t)^2.$$

Solving the quadratic, we get

$$\rho(x,t) = \frac{4t^2 - 4tx - 1 \pm \sqrt{8t^2 + 8tx + 1}}{8t^2}.$$

Choosing the positive sign so that $\lim_{(x,t)\to(0,0)} \rho(x,t) = \rho_0(0) = 1$, we obtain that provided no shock has formed,

$$\rho(x,t) = \begin{cases} 0, & x < -1 + t \text{ or } x > 1 + t \\ \frac{4t^2 - 4tx - 1 + \sqrt{8t^2+8tx+1}}{8t^2}, & \text{otherwise.} \end{cases} \tag{5}$$

(b) When $x_0 < -1$ the slope of the characteristics is $+1$ and when $x = 1$ the slope is -1, so characteristics will cross at some point and there will be a shock (see Figure). To find the time at which the shock forms, we can proceed by taking a partial x derivative of (1):

$$\rho_{xt} - 2\rho_x^2 + (1 - 2\rho)\rho_{xx} = 0.$$

Along characteristics, $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + [1 - 2\rho_0(x_0)]\frac{\partial}{\partial x}$, so $\rho_{xt} + (1-2\rho)\rho_{xx} = d\rho_x/dt$. Letting $y = \rho_x$, we get

$$\frac{dy}{dt} = 2y^2.$$

Solving this ODE with the initial conditions $y(0) = \rho_x(x_0)$ we get

$$y(t) = \begin{cases} \frac{-2x_0}{1+4tx_0}, & -1 < x < 1, \\ 0, & |x| \geq 1. \end{cases} \tag{6}$$

The time at which $y(t)$ diverges is $t = -1/(4x_0)$, which is minimized at $x_0 = -1$. Therefore the shock forms at $t = 1/4$. (You can also directly use the formula $t = -1/(2\min(\rho_0'))$.)

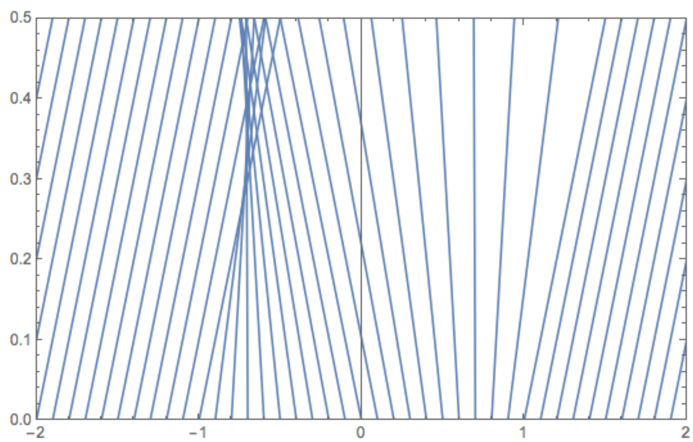

Figure 1: Characteristics given by $x = x_0 + [1 - 2\rho_0(x_0)]t$

2. **Heat Equation.** Consider the heat equation in an infinite rod

$$w_t = w_{xx}, \qquad -\infty < x < \infty, t > 0$$
$$w(x,0) = f(x),$$

where $f(x) \in C(\mathbb{R})$ is zero for $|x| > L$.

(a) (13 points) Show that

$$\int_{-\infty}^{\infty} w dx$$

is independent of time.

(b) (12 points) Show that

$$q_n(t) = \int_{-\infty}^{\infty} x^{2n} w dx$$

is a polynomial of degree n in t for $n \geq 0$.

Solution:

(a) We first show that $q_0 = \int_{-\infty}^{\infty} w dx$ is independent of time. Taking a time derivative, we get

$$\frac{dq_0}{dt} = \int_{-\infty}^{\infty} w_t dx = \int_{-\infty}^{\infty} w_{xx} dx = w_x\Big|_{-\infty}^{\infty}.$$

The solution w and its spatial derivative w_x can be expressed in terms of the fundamental solution as

$$w = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y) dy, \tag{7}$$
$$w_x = \int_{-\infty}^{\infty} \frac{-2(x-y)}{4t\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} f(y) dy, \tag{8}$$

and since $f(y) = 0$ for $|y| > L$, we have that for $x > L$ and larger than the value at which $(x-L)\exp(-(x-L)^2/4t)$ is maximized

$$|w_x| \leq \int_{-\infty}^{\infty} \frac{2|x-y|}{4t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} |f(y)| dy < \frac{2(x+L)}{4t} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-L)^2}{4t}} \int_{-L}^{L} |f(y)| dy.$$

Since $f \in C([-L,L])$ and the function multiplying the integral goes to zero as $x \to \infty$ for any fixed t, $\lim_{x\to\infty} |w_x| = 0$. Similarly, $\lim_{x\to-\infty} |w_x| = 0$, so

$$\frac{dq_0}{dt} = 0,$$

and $q_0(t)$ is a constant, i.e., a polynomial of degree 0.

(b) We proceed by induction. The case $n = 0$ was shown in (a). For the inductive step, assume $q_n(t)$ is a polynomial of degree n in t. Now let's calculate $\frac{dq_{n+1}}{dt}$:

$$\frac{dq_{n+1}}{dt} = \int_{-\infty}^{\infty} x^{2n+2} w_t dx = \int_{-\infty}^{\infty} x^{2n+2} w_{xx} dx.$$

Integrating by parts twice,

$$\frac{q_{n+1}}{dt} = x^{2n+2} w_x\Big|_{-\infty}^{\infty} - (2n+2)x^{2n+1} w\Big|_{-\infty}^{\infty} + (2n+1)(2n+2) \int_{-\infty}^{\infty} x^{2n} w dx.$$

Like before, one can show that $x^{2n+2}w_x\Big|_{-\infty}^{\infty} - (2n+2)x^{2n+1}w\Big|_{-\infty}^{\infty} = 0$, and so

$$\frac{q_{n+1}}{dt} = (2n+1)(2n+2)q_n,$$

which using the induction hypothesis is a polynomial of degree n in t. Integrating with respect to t, we find that q_{n+1} is a polynomial of degree $n+1$ in t.

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3. **Wave Equation.** Consider the following initial boundary value problem

$$u_{tt} = c^2 u_{xx}, \qquad x > 0, \quad t > 0,$$
$$u(x,0) = 0, \quad u_t(x,0) = g(x),$$
$$u_x(0,t) = 0.$$

(a) (10 points) Find all solutions to the above IBVP that lie in $x > 0$, $t > 0$.
Solution: By d'Alembert's approach, we know for $x > ct$,

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy,$$

but for $x < ct$, consider the interaction of the initial condition with the boundary. To do so, note the solutions take the form $u(x,t) \equiv F(x+ct) + G(x-ct)$, so $F(x) + G(x) = 0$ and $cF'(x) - cG'(x) = g(x)$ and $F'(ct) + G'(-ct) = F'(x) + G'(-x) = 0$. Integrating the 2nd and 3rd expressions yields

$$F(x) - G(x) = \frac{1}{c}\int_0^x g(y)dx + F(0) - G(0)$$
$$F(x) - G(-x) = F(0) - G(0).$$

Applying the 1st expression ($G(x) = -F(x)$) to integrated 2nd expression yields

$$F(x) = \frac{1}{2c}\int_0^x g(y)dy + F(0).$$

Rearranging the last expression then implies

$$G(x) = F(-x) - 2F(0) = \frac{1}{2c}\int_0^{-x} g(y)dy - F(0).$$

Thus, for $x < ct$,

$$u(x,t) = F(x+ct) + G(x-ct) = \frac{1}{2c}\left[\int_0^{x+ct} g(y)dy + \int_0^{ct-x} g(y)dy\right].$$

(b) (10 points) Using an energy functional, show if a solution to the IBVP exists, it is unique, given adequate assumptions. State these needed assumptions.
Solution: Define the following energy functional $E(t) \equiv \frac{1}{2}\int_0^\infty u_t^2(x,t) + c^2 u_x^2(x,t)dx$ and assume there are two solutions to the IBVP, u and v and take $w \equiv u - v$. First, note by linearity, w solves the same IBVP:

$$w_{tt} = c^2 w_{xx}, \quad w(x,0) \equiv w_t(x,0) \equiv 0, \quad w_x(0,t) \equiv 0.$$

Thus, passing $w(x,t)$ through the energy functional and differentiating with respect to time, we obtain

$$E'(t) \equiv \frac{d}{dt}\frac{1}{2}\int_0^\infty w_t^2(x,t) + c^2 w_x^2(x,t)dx = \int_0^\infty w_t(x,t)w_{tt}(x,t)dx + c^2\int_0^\infty w_{xt}(x,t)w_x(x,t)dx$$
$$= \int_0^\infty w_t(x,t)[w_{tt}(x,t) - c^2 w_{xx}(x,t)]dx = 0,$$

following from integration by parts and the IBVP for w. Thus, since $E(0) = 0$ by initial conditions, $E(t) \equiv 0$, which implies $w_t(x,t) \equiv w_x(x,t) \equiv 0$ as long as $w \in C^1$, bounded, and integrable. This implies $w \equiv 0$ since w vanishes on the boundaries, implying $u \equiv v$, indicating uniqueness.

(c) (5 points) Determine the region of influence of the segment $x \in [1,2]$ of the initial condition function $g(x)$. You may draw this in the upper right quadrant or write corresponding inequalities with respect to x and t.
Solution: The region of influence is bounded by the lines $t = 0$, $x = 0$, $1 - x = ct$, and $x - 2 = ct$

4. **Poisson's Equation/Green's Functions.**

(a) (8 points) Consider Poisson's equation with homogeneous Dirichlet boundary conditions in the half 2D ball:

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), \qquad && \mathbf{x} \in \Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 > 0 \; \& \; \|\mathbf{x}\| < 1\}, \\ u(\mathbf{x}) &= g(\mathbf{x}), && \mathbf{x} \in \partial\Omega. \end{aligned}$$

Determine the associated Green's function $G(\mathbf{x}, \mathbf{y})$ in terms of the fundamental solution $\Phi(|\mathbf{x}|) = \frac{1}{2\pi} \log |\mathbf{x}|$, and write the solution to the above boundary value problem.
Solution: Define $\tilde{\mathbf{x}} = (x_1, -x_2)$, $\hat{\mathbf{x}} = \mathbf{x}/\|\mathbf{x}\|^2$, and $\mathbf{x}^* = \tilde{\mathbf{x}}/\|\mathbf{x}\|^2$ then

$$G(\mathbf{x}, \mathbf{y}) = \Phi(|\mathbf{x} - \mathbf{y}|) - \Phi(|\tilde{\mathbf{x}} - \mathbf{y}|) - \Phi(\|\mathbf{x}\| \cdot |\hat{\mathbf{x}} - \mathbf{y}|) + \Phi(\|\mathbf{x}\| \cdot |\mathbf{x}^* - \mathbf{y}|).$$

The solution to the boundary value problem is then

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) d\mathbf{y} - \int_{\partial\Omega} \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{y}) g(\mathbf{y}) dS_{\mathbf{y}}.$$

(b) (8 points) Use the maximum principle to prove the uniqueness of the solution in part (a).
Solution: Assume u and v both solve the BVP, then $w = u - v$ solves

$$\begin{aligned} -\Delta w(\mathbf{x}) &= 0, \quad && \mathbf{x} \in \Omega \\ w(\mathbf{x}) &= 0, && \mathbf{x} \in \partial\Omega. \end{aligned}$$

By the maximum principle, since Ω is simply connected, any smooth solution to the BVP has its maximum and minimum on $\partial\Omega$ which is zero throughout, so $w \equiv 0$, implying $u \equiv v$.

(c) (9 points) Assume $f \equiv 0$ in the above BVP and prove for any ball $B_r(\mathbf{x})$ of radius r in Ω:

$$u(\mathbf{x}) = \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) dS_{\mathbf{y}} \quad \text{for all } B_r(\mathbf{x}) \subset \Omega.$$

Hint: Show that the function $h(r) = \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) dS_{\mathbf{y}}$ is constant in r.

Solution: Let

$$h(r) := \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) dS_{\mathbf{y}} = \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} u(\mathbf{x} + r\mathbf{z}) dS_{\mathbf{z}}$$

Then

$$\begin{aligned} h'(r) &= \frac{1}{|\partial B_1(0)|} \int_{\partial B_1(0)} \mathbf{z} \cdot \nabla u(\mathbf{x} + r\mathbf{z}) dS_{\mathbf{z}} = \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} \frac{\mathbf{y} - \mathbf{x}}{r} \cdot \nabla u(\mathbf{y}) dS_{\mathbf{y}} \\ &= \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} \frac{\partial u}{\partial \nu} dS_{\mathbf{y}} = \frac{1}{|\partial B_r(\mathbf{x})|} \int_{B_r(\mathbf{x})} \Delta u(\mathbf{y}) d\mathbf{y} = 0, \end{aligned}$$

where ν is the unit normal and by the divergence theorem. This implies the given integral is constant for all r. In the limit as $r \to 0$, note $h(r) \to u(\mathbf{x})$ for a smooth function u.

5. **Separation of Variables.** Consider Laplace's equation in the sector $W = \{(r, \theta) : 1 < r < a, 0 < \theta < \alpha\} \subseteq \mathbb{R}^2$

$$\Delta u = 0, \qquad x \in W,$$

with boundary conditions

$$\begin{aligned} u(r,0) &= u(r,\theta_0) = 0, \\ u(1,\theta) &= 0, \\ u(a,\theta) &= f(\theta). \end{aligned}$$

(a) (17 points) Find a formal solution $u(r,t)$ of the above boundary value problem in terms of f.

(b) (8 points) Find nontrivial conditions on g that guarantee that the solution you found is a classical solution u in $\mathcal{C}^2(\bar{W})$.

Solution:

(a) In polar coordinates, we have

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

Using separation of variables we set $u(r,t) = R(r)\Theta(\theta)$ and obtain

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\ddot{\Theta} = 0,$$

where prime and dot are derivatives with respect to r and θ, respectively. Separating variables we get

$$r^2 R'' + rR' = aR, \tag{9}$$
$$\ddot{\Theta} = -a\Theta. \tag{10}$$

First consider the angular equation. If $a < 0$ we get $\Theta(\theta) = Ae^{\sqrt{-a}\theta} + Be^{-\sqrt{-a}\theta}$, and from the boundary conditions $\Theta(0) = \Theta(\alpha) = 0$ we get $A = B = 0$. If $a = 0$ we get $\Theta(\theta) = A + B\theta$ and again we obtain $A = B = 0$. To avoid a trivial solution, a must be positive, and we get $\Theta(\theta) = A\sin(\sqrt{a}\theta) + B\cos(\sqrt{a}\theta)$. The boundary condition $\Theta(0) = 0$ gives $B = 0$, and $\Theta(\alpha) = 0$ gives $\sqrt{a} = 2n\pi/\alpha$, with $n \in \mathbb{Z}$. So we obtain the solutions

$$\sin\left(\frac{2\pi n\theta}{\alpha}\right), \qquad n \in \mathbb{Z}. \tag{11}$$

Now consider the radial equation. We have $r^2R'' + rR' - aR = 0$. Inserting the ansatz $R = r^\mu$ we get $\mu(\mu-1) + \mu - a = 0$, and so $\mu = \pm\sqrt{a} = \pm 2n\pi/\alpha$. Therefore we obtain the solutions

$$C_n r^{2n\pi/\alpha} + D_n r^{-2n\pi/\alpha}.$$

Now we construct the general solution as

$$u(r,\theta) = \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[C_n r^{2n\pi/\alpha} + D_n r^{-2n\pi/\alpha}\right]. \tag{12}$$

(Note that $n = 0$ does not contribute and $n < 0$ are redundant).

Now we need to choose C_n and D_n to satisfy the boundary conditions at $r = 1$ and $r = a$. At $r = 1$

$$u(1,\theta) = 0 = \sum_{n=1}^{\infty} \sin\left(\frac{2\pi n\theta}{\alpha}\right)[C_n + D_n].$$

Since this is true for all θ in $[0,\alpha]$, by orthogonality of $\{\sin\left(\frac{2\pi n\theta}{\alpha}\right)\}_{n\in\mathbb{N}}$ in $[0,\alpha]$ we must have

$$C_n + D_n = 0.$$

Therefore the solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[r^{2n\pi/\alpha} - r^{-2n\pi/\alpha}\right]. \tag{13}$$

Finally, we use the boundary condition at $r = a$ to find the coefficients C_n:

$$u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[a^{2n\pi/\alpha} - a^{-2n\pi/\alpha}\right]. \tag{14}$$

Multiplying by $\sin\left(\frac{2\pi k\theta}{\alpha}\right)$ and integrating from 0 to α we get

$$C_k = \left[a^{2k\pi/\alpha} - a^{-2k\pi/\alpha}\right]^{-1} \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\sin\left(\frac{2\pi k\theta}{\alpha}\right)d\theta. \tag{15}$$

(b) It is enough that $f(0) = f(\alpha)$, $f''(0) = f''(\alpha)$, $f \in C^4(\bar{W})$ for the solution to be a classical solution. To show this, we need to show that the solution (12) with coefficients (15) can be differentiated termwise as needed. We have

$$\begin{aligned}
u_{\theta\theta} &= -\sum_{n=1}^{\infty} C_n \left(\frac{2\pi n}{\alpha}\right)^2 \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[r^{2n\pi/\alpha} - r^{-2n\pi/\alpha}\right],\\
u_r &= \sum_{n=1}^{\infty} C_n \left(\frac{2\pi n}{\alpha}\right) \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[r^{2n\pi/\alpha-1} + r^{-2n\pi/\alpha-1}\right],\\
u_{rr} &= \sum_{n=1}^{\infty} C_n \left(\frac{2\pi n}{\alpha}\right) \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[\left(\frac{2\pi n}{\alpha} - 1\right) r^{2n\pi/\alpha-2} - \left(\frac{2\pi n}{\alpha} + 1\right) r^{-2n\pi/\alpha-2}\right].
\end{aligned}$$

If these series converge absolutely and uniformly, we can differentiate termwise. To show this, we use Weierstrass' M-test, and note that for $1 < r < a$

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left|C_n \left(\frac{2\pi n}{\alpha}\right)^2 \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[r^{2n\pi/\alpha} - r^{-2n\pi/\alpha}\right]\right| \le \sum_{n=1}^{\infty} |C_n| \left(\frac{2\pi n}{\alpha}\right)^2 \left[a^{2n\pi/\alpha} + 1\right],\\
&\sum_{n=1}^{\infty} \left|C_n \left(\frac{2\pi n}{\alpha}\right) \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[r^{2n\pi/\alpha-1} + r^{-2n\pi/\alpha-1}\right]\right| \le \sum_{n=1}^{\infty} |C_n| \left(\frac{2\pi n}{\alpha}\right) \left[a^{2n\pi/\alpha-1} + 1\right],\\
&\sum_{n=1}^{\infty} \left|C_n \left(\frac{2\pi n}{\alpha}\right) \sin\left(\frac{2\pi n\theta}{\alpha}\right)\left[\left(\frac{2\pi n}{\alpha} - 1\right) r^{2n\pi/\alpha-2} - \left(\frac{2\pi n}{\alpha} + 1\right) r^{-2n\pi/\alpha-2}\right]\right| \le \\
&\sum_{n=1}^{\infty} |C_n| \left(\frac{2\pi n}{\alpha}\right)\left(\frac{2\pi n}{\alpha} + 1\right)\left[a^{2n\pi/\alpha-2} + 1\right].
\end{aligned} \tag{16}$$

Therefore, it is enough to show that $|C_n|(a^{2n\pi/\alpha-2}+1) \leq \frac{M}{n^4}$ for some constant M. Integrating (15) by parts twice and using $f(0)=f(\alpha)$ we get

$$C_k\left[a^{2k\pi/\alpha}-a^{-2k\pi/\alpha}\right] = -\frac{2}{\alpha}\left(\frac{\alpha}{2\pi k}\right)^2 \int_0^\alpha f''(\theta)\sin\left(\frac{2\pi k\theta}{\alpha}\right)d\theta. \tag{17}$$

Integrating by parts twice again and using $f''(0)=f''(\alpha)$ we get

$$C_k\left[a^{2k\pi/\alpha}-a^{-2k\pi/\alpha}\right] = \frac{2}{\alpha}\left(\frac{\alpha}{2\pi k}\right)^4 \int_0^\alpha f''''(\theta)\sin\left(\frac{2\pi k\theta}{\alpha}\right)d\theta. \tag{18}$$

Using $f \in C^4(\bar{W})$, we find, letting $E = \max_{[0,\alpha]}|f''''|$

$$|C_k| \leq \left[a^{2k\pi/\alpha}-a^{-2k\pi/\alpha}\right]^{-1}\frac{2}{\alpha}\left(\frac{\alpha}{2\pi k}\right)^4 \alpha E, \tag{19}$$

and so

$$|C_n|(a^{2n\pi/\alpha-2}+1) \leq \frac{a^{2n\pi/\alpha-2}+1}{a^{2n\pi/\alpha}-a^{-2n\pi/\alpha}}\frac{2}{\alpha}\left(\frac{\alpha}{2\pi n}\right)^4 \alpha E \leq \frac{M}{n^4}, \tag{20}$$

and the formal solution can be differentiated termwise as required.