Preliminary Exam
Partial Differential Equations
10:00 AM - 1:00 PM, Friday, Aug. 28, 2020
Room: ECCR 244

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Quasilinear first order equations.** The density of cars $\rho(x,t)$ in a traffic model satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left[\rho(1-\rho)\right] = 0, \qquad t > 0, -\infty < x < \infty$$

with initial conditions

$$\rho(x,0) = \rho_0(x) = \begin{cases} 1 - x^2, & -1 < x < 1, \\ 0, & |x| \geq 1. \end{cases}$$

(a) (15 points) Find $\rho(x,t)$ for times t less than the time at which a shock forms.

(b) (10 points) Find the time at which a shock forms.

2. **Heat Equation.** Consider the heat equation in an infinite rod

$$\begin{aligned} w_t &= w_{xx}, & -\infty < x < \infty, t > 0 \\ w(x,0) &= f(x), & \end{aligned}$$

where $f(x) \in C(\mathbb{R})$ is zero for $|x| > L$.

(a) (13 points) Show that

$$\int_{-\infty}^{\infty} w dx$$

is independent of time.

(b) (12 points) Show that

$$q_n(t) = \int_{-\infty}^{\infty} x^{2n} w dx$$

is a polynomial of degree n in t for $n \geq 0$.

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3. **Wave Equation.** Consider the following initial boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, \qquad x > 0, \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = g(x), \\ u_x(0,t) &= 0. \end{aligned}$$

(a) (10 points) Find all solutions to the above IBVP that lie in $x > 0$, $t > 0$.
(b) (10 points) Using an energy functional, show if a solution to the IBVP exists, it is unique, given adequate assumptions. State these needed assumptions.
(c) (5 points) Determine the region of influence of the segment $x \in [1, 2]$ of the initial condition function $g(x)$. You may draw this in the upper right quadrant or write corresponding inequalities with respect to x and t.

4. **Poisson's Equation/Green's Functions.**
(a) (8 points) Consider Poisson's equation with homogeneous Dirichlet boundary conditions in the half 2D ball:

$$\begin{aligned} -\Delta u(\mathbf{x}) &= f(\mathbf{x}), \qquad && \mathbf{x} \in \Omega = \{\mathbf{x} \in \mathbb{R}^2 \mid x_2 > 0 \ \& \ ||\mathbf{x}|| < 1\}, \\ u(\mathbf{x}) &= g(\mathbf{x}), && \mathbf{x} \in \partial\Omega. \end{aligned}$$

Determine the associated Green's function $G(\mathbf{x}, \mathbf{y})$ in terms of the fundamental solution $\Phi(|\mathbf{x}|) = \frac{1}{2\pi} \log |\mathbf{x}|$, and write the solution to the above boundary value problem.
(b) (8 points) Use the maximum principle to prove the uniqueness of the solution in part (a).
(c) (9 points) Assume $f \equiv 0$ in the above BVP and prove for any ball $B_r(\mathbf{x})$ of radius r in Ω:

$$u(\mathbf{x}) = \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) dS_{\mathbf{y}} \quad \text{for all } B_r(\mathbf{x}) \subset \Omega.$$

Hint: Show that the function $h(r) = \frac{1}{|\partial B_r(\mathbf{x})|} \int_{\partial B_r(\mathbf{x})} u(\mathbf{y}) dS_{\mathbf{y}}$ is constant in r.

5. **Separation of Variables.** Consider Laplace's equation in the sector $W = \{(r, \theta) : 1 < r < a, 0 < \theta < \alpha\} \subseteq \mathbb{R}^2$

$$\Delta u = 0, \qquad x \in W,$$

with boundary conditions

$$\begin{aligned} u(r, 0) &= u(r, \theta_0) = 0, \\ u(1, \theta) &= 0, \\ u(a, \theta) &= f(\theta). \end{aligned}$$

(a) (17 points) Find a formal solution $u(r, t)$ of the above boundary value problem in terms of f.

(b) (8 points) Find nontrivial conditions on g that guarantee that the solution you found is a classical solution u in $\mathcal{C}^2(\bar{W})$.