Preliminary Exam
Partial Differential Equations
9:00AM – 12:00PM, 22, Aug 2023

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Heat equation.**

   (a) (13 points) Consider the following initial boundary value problem on the annulus defined by $\Omega \equiv \{(r,\theta) \mid r \in (1,2) \ \&\ \theta \in [0,2\pi)\}$:

$$\begin{aligned} u_t &= \Delta u, && (r,\theta)\in\Omega, && t\in(0,\infty),\\ u(1,\theta,t) &= u(2,\theta,t) = 1, && \theta\in[0,2\pi), && t\in(0,\infty),\\ u(r,\theta,0) &= r^2-3r+3, && r\in(1,2), && \theta\in[0,2\pi). \end{aligned}$$

   Assuming existence of a classical solution $u(r,\theta,t)$, show that $u(r,\theta,t) > \frac{3}{4}$ on $\Omega\times\{t>0\}$.
   Solution: The minimum on $r=1,2$ is $u=1$ and on $t=0$ is $u(3/2,\theta,t)=3/4$. Define

$$U_T = \{(r,\theta,t)\in\Omega\times(0,T]\}, \quad \text{for any } T\in(0,\infty)$$

   The weak minimum principle implies $\min_{\bar{U}_T} u \equiv 3/4$. By the strong minimum principle, if $\min_{U_T} u \equiv 3/4$ then $u\equiv 3/4$ on U_T, but this cannot be since $u(r,\theta,0) = r^2-3r+3$, so $u > 3/4$ for all U_T and any $T\in(0,\infty)$.

   (b) (12 points) Show the solution of the system in part (a) is unique.
   Solution: Assume two solutions u and v, then $w = u - v$ satisfies

$$\begin{aligned} w_t &= \Delta w, && (r,\theta)\in\Omega, && t\in(0,\infty),\\ w(1,\theta,t) &= w(2,\theta,t) = 0, && \theta\in[0,2\pi), && t\in(0,\infty),\\ w(r,\theta,0) &= 0, && r\in(1,2), && \theta\in[0,2\pi). \end{aligned}$$

   The maximum and minimum principle ensure $\max_{U_T} w \equiv \max_{U_T} w \equiv 0$ for any U_T as defined in (a). Thus $w\equiv 0$ for any U_T so $u\equiv v$.

2. **Wave equation.**

(a) (10 points) Consider the following initial boundary value problem

$$
\begin{aligned}
u_{tt} &= \Delta u, && \mathbf{x} \in \Omega, \quad t \in (0,\infty),\\
u(\mathbf{x},0) &= f(\mathbf{x}), \quad u_t(\mathbf{x},0) = g(\mathbf{x}), && \mathbf{x} \in \Omega,\\
\hat{n}\cdot\nabla u + a(\mathbf{x})\frac{\partial u}{\partial t} &= 0, && \mathbf{x} \in \partial\Omega,
\end{aligned}
$$

where $\hat{n}\cdot\nabla u$ is the normal derivative, Ω is a bounded domain in $\mathbb{R}^n$, and $a(\mathbf{x}) \geq 0$. Assume that u is a classical solution, and define the energy $E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 d\mathbf{x}$, and show $E(t) \leq E(0)$ for $t \geq 0$.

Solution: We prove this by showing the energy's time derivative is non-positive

$$
\begin{aligned}
E'(t) &= \int_\Omega (u_t u_{tt} + \nabla u \cdot \nabla u_t) d\mathbf{x} = \int_\Omega u_t u_{tt} d\mathbf{x} + \int_{\partial\Omega} u_t \left[\hat{n}\cdot\nabla u\right] ds - \int_\Omega u_t \Delta u d\mathbf{x}\\
&= \int_\Omega u_t(u_{tt} - \Delta u) d\mathbf{x} - \int_{\partial\Omega} a(\mathbf{x}) u_t^2 d\mathbf{x} = -\int_{\partial\Omega} a(\mathbf{x}) u_t^2 d\mathbf{x} \leq 0
\end{aligned}
$$

where we have used Green's first identity, the boundary condition, and the non-negativity of a and u_t^2 (and thus their product).

(b) (15 points) With the aid of the energy $E(t)$ defined in part (a), prove the uniqueness of classical solutions to the initial boundary value problem.

Solution: Suppose u and v are two solutions of the IBVP and let $w \equiv u - v$, then

$$
\begin{aligned}
w_{tt} &= \Delta w, && \mathbf{x} \in \Omega, \quad t \in (0,\infty),\\
w(\mathbf{x},0) &= w_t(\mathbf{x},0) = 0, && \mathbf{x} \in \Omega,\\
\frac{\partial w}{\partial n} + a(\mathbf{x})\frac{\partial w}{\partial t} &= 0, && \mathbf{x} \in \partial\Omega.
\end{aligned}
$$

Then, we have $E_w(0) = \frac{1}{2}\int_\Omega w_t^2 + |\nabla w|^2 d\mathbf{x} = 0$, and by our result in (a), $E_w(t) \leq E_w(0) = 0$ but also $E_w(t) \geq 0$ due to the non-negativity of the integrand. Thus, $E_w(t) \equiv 0$ for all $t \geq 0$, so $w_t \equiv 0$ and $\frac{\partial w}{\partial x_j} = 0$ for all $j = 1, ..., n$, so $w(\mathbf{x}, t)$ is constant but by the initial conditions this constant is zero so $u \equiv v$.

3. **Method of characteristics.** Consider the PDE

$$xuu_x + yuu_y = xy \ ,$$

on the domain $\Omega = \{(x, y) : x \geq 1, y \in \mathbb{R}\}$, with the initial condition $u(1, y) = \tanh(y)$.

(a) Write out the characteristic equations for this PDE

(b) Solve these ODEs [Hint: You might find it helpful to rewrite the characteristic equations for (y, u) as functions of x , i.e for dy/dx and du/dx].

(c) Find the expression for $u(x, y)$. (Make sure you choose the proper sign for any square roots!)

(d) Does this solution exist for all points in Ω?

Solution:

(a) The characteristic equations are

$$\frac{dx}{d\tau} = xu$$
$$\frac{dy}{d\tau} = yu$$
$$\frac{du}{d\tau} = xy$$

(b) Using x as the independent variable instead gives

$$\frac{dy}{dx} = \frac{\frac{dy}{d\tau}}{\frac{dx}{d\tau}} = \frac{y}{x}$$
$$\frac{du}{dx} = \frac{y}{u}$$

Solving these with the initial condition $y(1) = s$, and $u(1) = \tanh(s)$, gives

$$y = sx$$
$$u^2 = s(x^2 - 1) + \tanh^2(s)$$

(c) So we can solve for $s = y/x$ and get

$$u(x, y) = \operatorname{sgn}(y)\sqrt{xy - \tfrac{y}{x} + \tanh^2(\tfrac{y}{x})}$$

Note that we added a sign outside the square root to get the proper $u(1, y)$ when $y < 0$.

(d) Note that if $y > 0$ and $x \geq 1$ the argument of the $\sqrt{}$ is always positive. However, when $y < 0$ there are problems when we hit the solutions of

$$\frac{y}{x} = -\frac{\tanh^2(y/x)}{x^2 - 1}$$

This transcendental equation does have solutions, which correspond to shocks in the PDE.

4. **Poisson's Equation/Green's Functions.**

(a) (10 points) State and prove the weak maximum principle for Laplace's equation:

$$\Delta u = 0, \qquad \mathbf{x} \in \Omega,$$
$$u = g, \qquad \mathbf{x} \in \partial\Omega, \qquad u \text{ is bounded and } C^2(\Omega) \cap C(\bar{\Omega}).$$

Solution: Take $\max_{\mathbf{x}\in\bar{\Omega}\setminus\Omega} u(\mathbf{x}) =: M$. Taking $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon|\mathbf{x}|^2$ for any $\epsilon > 0$, if we assume $v(\mathbf{x})$ obtains a maximum at $\mathbf{x}_0 \in \Omega$, then $\nabla v(\mathbf{x}_0) = 0$ and $\Delta v(\mathbf{x}_0) < 0$. However,

$$\Delta v(\mathbf{x}) = \Delta u(\mathbf{x}) + 2n\epsilon > 0,$$

which is a contradiction, so $v(\mathbf{x}) = u(\mathbf{x}) + \epsilon|\mathbf{x}|^2 \leq M + \epsilon C$ where $C = \max_{\mathbf{x}\in\Omega} |\mathbf{x}|^2$. Since this is true for any $\epsilon > 0$, then $u(\mathbf{x}) \leq \max_{\mathbf{x}\in\bar{\Omega}\setminus\Omega} u(\mathbf{x})$.

(b) (5 points) For $u(r, \theta)$ defined on $\Omega \equiv B(0, 1) \subset \mathbb{R}^2$ and $u(1, \theta) = g(\theta) = 2 + \cos(\theta)$ on $\theta \in [0, 2\pi)$, determine $u(0, \theta)$. Justify your answer, stating any needed theorems.

(Hint: You need not solve the boundary value problem.)

Solution: Applying the mean value property

$$u(\mathbf{x}) = \fint_{\partial B(\mathbf{x},R)} u(\mathbf{y})\mathrm{d}S(\mathbf{y}).$$

Thus, if we draw a circle around the origin, right at the boundary, we have

$$u(0, \theta) = \frac{1}{2\pi}\int_0^{2\pi} 2 + \cos(\theta) d\theta = 2.$$

(c) (10 points) Consider Poisson's equation on the half-disc:

$$\Delta u = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega \equiv \left\{\mathbf{x} \in \mathbb{R}^2 \mid x_1 > 0 \ \& \ |\mathbf{x}| < 1\right\},$$
$$u = 0, \qquad \mathbf{x} \in \partial\Omega, \qquad u \text{ is bounded and } C^2(\Omega) \cap C(\bar{\Omega}).$$

Determine the associated Green's function $G_S(\mathbf{x}, \mathbf{y})$ in terms of the fundamental solution to the two-dimensional Laplace equation, $\Phi(\mathbf{x}) = -\frac{1}{2\pi}\log|\mathbf{x}|$, and write the solution to the above boundary value problem, showing it satisfies $u(\mathbf{x}) = 0$ on $\mathbf{x} \in \partial\Omega$.

Solution: Define $\tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|^2$ and $\mathbf{x}_H = (-x_1, x_2)$, then using method of images:

$$G_S(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x}_H - \mathbf{y}) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) + \Phi(|\mathbf{x}|(\tilde{\mathbf{x}}_H - \mathbf{y})),$$

where Φ is the fundamental solution to Laplace's equation, such that $\Delta_{\mathbf{x}}\Phi(\mathbf{x} - \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y})$. When $\mathbf{x} = (0, x_2)$, then we have $\mathbf{x}_H = \mathbf{x}$ and $\tilde{\mathbf{x}}_H = \tilde{\mathbf{x}}$, so

$$G_S(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x} - \mathbf{y}) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) + \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})) = 0,$$

and when $|\mathbf{x}| = 1$ with $\mathbf{x} \in \partial\Omega$, then $\tilde{\mathbf{x}} = \mathbf{x}$ and $\tilde{\mathbf{x}}_H = \mathbf{x}_H$ and $|\mathbf{x}_H| = 1$, so

$$G_S(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\mathbf{x}_H - \mathbf{y}) - \Phi((\mathbf{x} - \mathbf{y})) + \Phi((\mathbf{x} - \mathbf{y})) = 0.$$

The solution to the BVP is then

$$u(\mathbf{x}) = \int_\Omega G_S(\mathbf{x}, \mathbf{y})f(\mathbf{y})d\mathbf{y},$$

so when $\mathbf{x} \in \partial\Omega$ then

$$u(\mathbf{x}) = \int_\Omega 0 \cdot f(\mathbf{y})d\mathbf{y} = 0.$$

5. **Separation of Variables.** Solve the forced wave equation

$$u_{tt} = c^2 u_{xx} + \cos(x)\cos(ct)$$

on the domain $\Omega = \{(x,t) : t > 0\ ,\ x \in (-\pi, \pi)\}$ with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 3\cos(2x)$$

and periodic boundary conditions

$$u(-\pi,t) = u(\pi,t) \quad \text{and} \quad u_x(-\pi,t) = u_x(\pi,t).$$

Solution: To accommodate the forcing let $u(x,t) = \phi(x,t) + f(t)\cos(x)$, where ϕ solves the unforced case, and we have the ODE

$$f'' = c^2 f + \cos(ct)$$

This ODE has a resonance and can be solved by assuming $f(t) = at\sin(ct)$, which gives

$$f'' - c^2 f = 2ac\cos(ct)$$

so we take $a = 1/2c$. Now we must simply solve $\phi'' = -c^2\phi$ on the periodic domain. Using $\phi(x,t) = X(x)T(t)$ gives the usual form

$$\phi(x,t) = A_0 + \sum_{n=1}^{\infty}(A_n\cos(nx) + B_n\sin(nx))(C_n\cos(cnt) + D_n\sin(cnt))$$

We now have initial conditions

$$\begin{aligned}\phi(x,0) &= u(x,0) = 0,\\ \phi_t(x,0) &= u_t(x,0) - f_t(0)\cos(x) = 3\cos(2x)\end{aligned}$$

So we find that $A_2 D_2(2c) = 3$ and the remaining coefficents are zero. Thus we have

$$u(x,t) = \frac{t}{2c}\cos(x)\sin(ct) + \frac{3}{2c}\cos(2x)\sin(2ct)$$