Preliminary Exam
Partial Differential Equations
9:00AM – 12:00PM, 22, Aug 2023

Student ID (do NOT write your name):

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#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

There are five problems. **Solve four of the five problems.**
Each problem is worth 25 points.
A sheet of convenient formulae is provided.

1. **Heat equation.**

   (a) (13 points) Consider the following initial boundary value problem on the annulus defined by $\Omega \equiv \{(r,\theta) \mid r \in (1,2) \ \&\ \theta \in [0,2\pi)\}$:

$$
\begin{aligned}
u_t &= \Delta u, && (r,\theta) \in \Omega, && t \in (0,\infty),\\
u(1,\theta,t) &= u(2,\theta,t) = 1, && \theta \in [0,2\pi), && t \in (0,\infty),\\
u(r,\theta,0) &= r^2 - 3r + 3, && r \in (1,2), && \theta \in [0,2\pi).
\end{aligned}
$$

   Assuming existence of a classical solution $u(r,\theta,t)$, show that $u(r,\theta,t) > \frac{3}{4}$ on $\Omega \times \{t > 0\}$.

   (b) (12 points) Show the solution of the system in part (a) is unique.

2. **Wave equation.**

   (a) (10 points) Consider the following initial boundary value problem

$$
\begin{aligned}
u_{tt} &= \Delta u, && && \mathbf{x} \in \Omega, && t \in (0,\infty),\\
u(\mathbf{x},0) &= f(\mathbf{x}), && u_t(\mathbf{x},0) = g(\mathbf{x}), && \mathbf{x} \in \Omega,\\
\hat{n}\cdot\nabla u + a(\mathbf{x})\frac{\partial u}{\partial t} &= 0, && && \mathbf{x} \in \partial\Omega,
\end{aligned}
$$

   where $\hat{n}\cdot\nabla u$ is the normal derivative, Ω is a bounded domain in $\mathbb{R}^n$, and $a(\mathbf{x}) \geq 0$. Assume that u is a classical solution, and define the energy $E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 d\mathbf{x}$, and show $E(t) \leq E(0)$ for $t \geq 0$.

   (b) (15 points) With the aid of the energy $E(t)$ defined in part (a), prove the uniqueness of classical solutions to the initial boundary value problem.

3. **Method of characteristics.** Consider the PDE

$$xuu_x + yuu_y = xy \ ,$$

on the domain $\Omega = \{(x,y) : x \geq 1, y \in \mathbb{R}\}$, with the initial condition $u(1,y) = \tanh(y)$.

(a) Write out the characteristic equations for this PDE

(b) Solve these ODEs [Hint: You might find it helpful to rewrite the characteristic equations for (y,u) as functions of x , i.e for dy/dx and du/dx].

(c) Find the expression for $u(x,y)$. (Make sure you choose the proper sign for any square roots!)

(d) Does this solution exist for all points in Ω?

4. **Poisson's Equation/Green's Functions.**

(a) (10 points) State and prove the weak maximum principle for Laplace's equation:

$$\begin{aligned} \Delta u &= 0, && \mathbf{x} \in \Omega, \\ u &= g, && \mathbf{x} \in \partial\Omega, \qquad u \text{ is bounded and } C^2(\Omega) \cap C(\bar{\Omega}). \end{aligned}$$

(b) (5 points) For $u(r,\theta)$ defined on $\Omega \equiv B(0,1) \subset \mathbb{R}^2$ and $u(1,\theta) = g(\theta) = 2 + \cos(\theta)$ on $\theta \in [0, 2\pi)$, determine $u(0,\theta)$. Justify your answer, stating any needed theorems.

(Hint: You need not solve the boundary value problem.)

(c) (10 points) Consider Poisson's equation on the half-disc:

$$\begin{aligned} \Delta u &= f(\mathbf{x}), && \mathbf{x} \in \Omega \equiv \left\{ \mathbf{x} \in \mathbb{R}^2 \mid x_1 > 0 \ \& \ |\mathbf{x}| < 1 \right\}, \\ u &= 0, && \mathbf{x} \in \partial\Omega, \qquad u \text{ is bounded and } C^2(\Omega) \cap C(\bar{\Omega}). \end{aligned}$$

Determine the associated Green's function $G_S(\mathbf{x}, \mathbf{y})$ in terms of the fundamental solution to the two-dimensional Laplace equation, $\Phi(\mathbf{x}) = -\frac{1}{2\pi} \log |\mathbf{x}|$, and write the solution to the above boundary value problem, showing it satisfies $u(\mathbf{x}) = 0$ on $\mathbf{x} \in \partial\Omega$.

5. **Separation of Variables.** Solve the forced wave equation

$$u_{tt} = c^2 u_{xx} + \cos(x)\cos(ct)$$

on the domain $\Omega = \{(x,t) : t > 0 \ , \ x \in (-\pi, \pi)\}$ with the initial conditions

$$u(x,0) = 0, \quad u_t(x,0) = 3\cos(2x)$$

and periodic boundary conditions

$$u(-\pi, t) = u(\pi, t) \quad \text{and} \quad u_x(-\pi, t) = u_x(\pi, t).$$