RBF Partition of Unity

February 28, 2018

Outline

Problem

- Solving partial differential equations (PDEs) or interpolating data with global radial basis functions (RBFs) can lead to large node sets
- Therefore the dependence of computational time as a function of the number of nodes (N) becomes important
- Using N nodes with Global RBF $\phi(|\mathbf{x}_i - \mathbf{x}_j|)$, **A** is a dense N by N matrix

$$A = \begin{bmatrix} \phi(|\mathbf{x}_0 - \mathbf{x}_0|) & \phi(|\mathbf{x}_0 - \mathbf{x}_1|) & \dots & \phi(|\mathbf{x}_0 - \mathbf{x}_{N-1}|) \\ \phi(|\mathbf{x}_1 - \mathbf{x}_0|) & \phi(|\mathbf{x}_1 - \mathbf{x}_1|) & \dots & \phi(|\mathbf{x}_0 - \mathbf{x}_{N-1}|) \\ \vdots & \vdots & \ddots & \vdots \\ \phi(|\mathbf{x}_{N-1} - \mathbf{x}_0|) & \phi(|\mathbf{x}_{N-1} - \mathbf{x}_1|) & \dots & \phi(|\mathbf{x}_{N-1} - \mathbf{x}_{N-1}|) \end{bmatrix}$$

Problem

- For a dense N by N matrix $\mathbf{A}$
- Getting weights requires solving a system of linear equation $\mathbf{A}\lambda = \mathbf{u}$
  - $\mathcal{O}(N^3)$ operation
- Applying the Differentiation Matrix (DM) is a matrix multiplication
  - $\mathcal{O}(N^2)$ operation
- The resulting methods are computational difficult for large node sets
- RBF Partition of Unity (RBF-PUM) methods look to avoid these issues
- An additional leading approach for attacking these issues is RBF-FD

Brief RBF-PUM history

- The idea of RBF-PUM was original proposed by Babuška and Melenk (1997)
- Used for interpolation with compactly supported RBFs by Wedland (2002)
- Further discussion in Fasshauer's book (2007)
- Explored for interpolating non uniform data sets in Cavoretto, De Rossi *et. al.* (2012-Present)
- Used to solve PDEs in E. Larsson *et. al.* (2012-present) and J. Ahlkrona and V. Shcherbakov (2017-present)

Partition of Unity

- Rather than taking N global nodes, the space is partitioned
- The partitions overlap, however the sum of their weights are unity throughout the whole space.
- The resulting matrix becomes sparse
- $\mathbf{A}\lambda = \mathbf{u}$ solution and matrix multiplication can be done significantly faster for sparse matrices.

Partition of Unity

Partition of Unity starts with a set of patches $\Omega_j, j = 1, ..., P$ that cover a domain Ω

$$\bigcup_{j=1}^{P} \Omega_j \supseteq \Omega$$

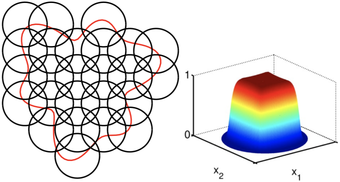


FIG. 3.1. *To the left, the red curve is the outline of the domain Ω_S and the black circles are the boundaries of the overlapping circular patches Ω_j, $j = 1, \ldots, P$. To the right, a partition of unity weight function w_j for one of the interior patches is shown.*

Partition of Unity

Each patch has a weight ω_j such that

$$\sum_{j=1}^{P} \omega_j(\mathbf{x}) = 1, \forall \mathbf{x} \in \Omega$$

$$\omega_j(\mathbf{x}) = \frac{\phi_j(\mathbf{x})}{\sum_{i=1}^{P} \phi(\mathbf{x})}$$

One set of generating functions $\phi(r)$ (Wendland functions)

$$\phi_j(r) = (4r + 1)(1 - r^4)_+$$

pertain to a patch with center $\mathbf{c}_j$ and radius ρ_j

$$\phi_j(\mathbf{x}) = \phi_j\left(\frac{|\mathbf{x} - \mathbf{c}_j|}{\rho_j}\right)$$

giving an interpolate from local interpolates $u_j(\mathbf{x})$

$$u(\mathbf{x}) = \sum_{j=1}^{P} \omega_j(\mathbf{x}) u_j(\mathbf{x})$$

RBF-PUM

To move from PUM to RBF-PUM one introduces a RBF $\Phi(r)$ such as a Gaussian

$$\Phi(r) = e^{-(\epsilon r)^2}$$

and a set of node points $\mathbf{x}_i$ where $u(\mathbf{x}_i)$ is known.
Using the global RBF tools one gets the local interpolates $u_j(\mathbf{x})$ such that

$$u_j(\mathbf{x}) = \sum_{i=1}^{n_j} \lambda_i^j \Phi(|\mathbf{x} - \mathbf{x}_i|)$$

with the λ_i^j obtained using Global RBF in the particular patch. (RBF-QR)

RBF-PUM

The same framework can be extended to a differential operator $\mathcal{L}$ such that

$$\mathcal{L}u(\mathbf{x}) = \sum_{j=1}^{P} \mathcal{L}(\omega_j(\mathbf{x})u_j(\mathbf{x}))$$

This allows for the use of RBF-PUM to solve PDEs

C-RBF-PUM

Global node set

$$X = \{\mathbf{x}_k\}_{k=1}^{N}$$

Consisting of interior nodes

$$X^i = \{\mathbf{x}_k \in X : \mathbf{x}_k \in \Omega\}$$

and boundary nodes

$$X^b = \{\mathbf{x}_k \in X : \mathbf{x}_k \in \partial\Omega\}$$

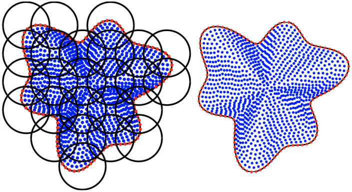


FIG. 3.2. *A global node set and patches in Ω_S (left), and evaluation points for collocation (same as node points) (right) for C-RBF-PUM.*

E. Larsson, V. Shcherbakov, and A. Heryudono (2017)

LS-RBF-PUM

- Start with identical node sets X_i with respect to a patch Ω_i
- Global set gives

$$X = \bigcup_{j=1}^{P} X_j$$

- The uniformity of X_i lead to only one linear system solve
- Decouples evaluation points $Y^i \subset Y$ and $Y^b \subset Y$

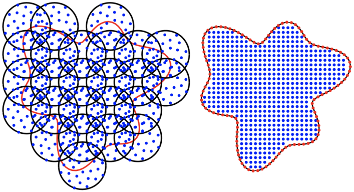
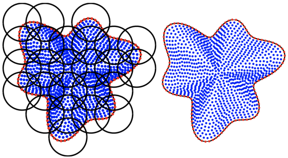


FIG. 3.3. *Patches with identically distributed local node sets covering the domain Ω_S (left), and least squares evaluation points on a Cartesian grid in the interior and uniform with respect to arc length on the boundary (right) for LS-RBF-PUM.*

E. Larsson, V. Shcherbakov, and A. Heryudono (2017)

Comparison



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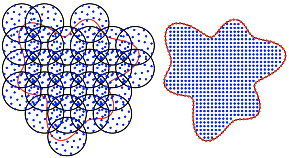

FIG. 3.3. Patches with identically distributed local node sets covering the domain Ω_S (left), and least squares evaluation points on a Cartesian grid in the interior and uniform with respect to arc length on the boundary (right) for LS-RBF-PUM.

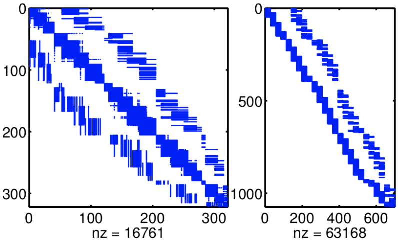

FIG. 3.4. The structure of the matrix L for a problem defined over Ω_S with $P = 24$ patches, box size $H = 0.6$, and overlap $\delta = 0.2$. For C-RBF-PUM (left), $13 \leq n_j \leq 42$, $h \approx 0.12$, and $N = 321$, and for LS-RBF-PUM (right), $n = 28$, $h \approx 0.14$, $N = 700$, $M = 1073$, and $\beta = M/N \approx 1.5$.

E. Larsson, V. Shcherbakov, and A. Heryudono (2017)

Comparison

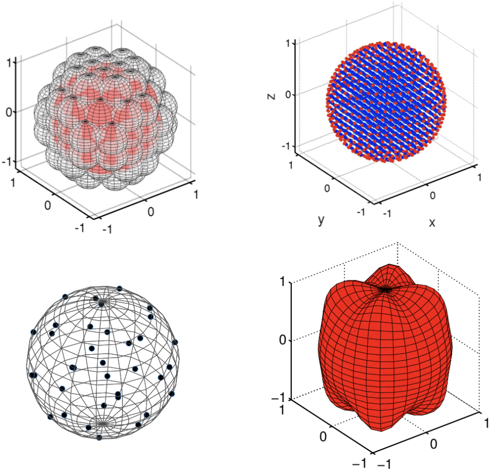

FIG. 5.10. *Patches on the spherical domain* Ω_U *(top left), quasi uniformly distributed least squares points inside* Ω_U *and on the surface* $\partial\Omega_U$ *(top right), a single patch (enlarged) with* $n = 35$ *local node points (bottom left), and the star shaped domain* Ω_Q *(bottom right).*

E. Larsson, V. Shcherbakov, and A. Heryudono (2017)

Patch Sets

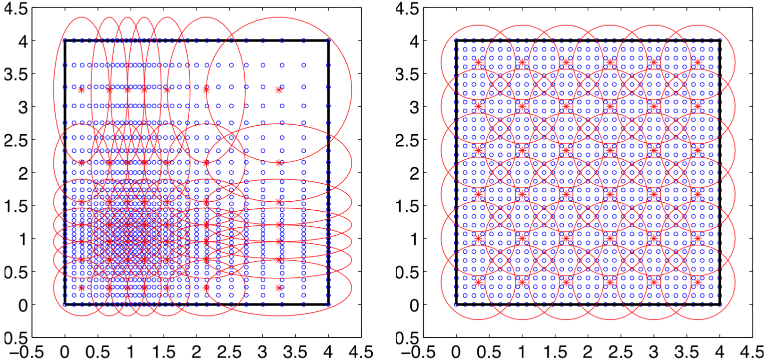

Fig. 12. Discretization of the square domain with oval and circle patches

A. Safdari-Vaighani, A.R.H. Heryudono, and E. Larsson (2015)

Conclusions

- RBF-PUM was proposed in 1997
- C-RBF-PUM and LS-RBF-PUM algorithms were presented
- Expended from 2D to 3D with spheres
- Non uniform patch sets have been shown
- The algorithms presented are by no means exhaustive

Questions

Thank you for your attention!