Preliminary Examination (Solutions): Partial Differential Equations
1 PM—4 PM, Jan. 12, 2023,
Newton Lab

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Student ID:____________________________________

There are five problems. **Solve four of the five problems and circle which four you choose in the grading key on the right.** Each problem is worth 25 points. Please start each problem on a new page. A sheet of convenient formulae is provided.

1. **(Solution methods)** Consider the Airy equation

$$u_t + u_{xxx} = 0, \quad x \in (0, 2\pi), \quad t \in (0, \infty)$$

with periodic boundary conditions $u(0,t) = u(2\pi, t)$, $u_x(0,t) = u_x(2\pi, t)$, and $u_{xx}(0,t) = u_{xx}(2\pi, t)$ for all $t \geq 0$, and initial condition $u(x, 0) = f(x)$. The initial condition f is assumed to be real-valued, 2π-periodic, and C^∞.

(a) Show that $M(t) = \int_0^{2\pi} u(x,t)dx$ remains constant in time.

(b) Let $H^k(t) = \int_0^{2\pi} \left[\frac{\partial^k}{\partial x^k} u(x,t)\right]^2 dx$, $k = 0, 1, 2, \ldots$ (H^0 is the L_2 norm of u). Show that H^0 and H^1 remain constant in time. Is this also true for the H^k with higher k?

(c) Give a series solution of the inhomogeneous Airy equation

$$\begin{aligned} u_t + u_{xxx} &= \cos(2x + \omega t), \quad 0 \leq x \leq 2\pi,\ t \geq 0 \\ u(x, 0) &= 0, \quad 0 \leq x \leq 2\pi \\ u(0,t) &= u(2\pi, t) \quad t \geq 0 \end{aligned}$$

Discuss the qualitative difference between solutions corresponding to $\omega = 8$ and $\omega \neq 8$.
Useful fact: If $g(x)$ is real and periodic, then its complex Fourier coefficients g_k satisfy $g_{-k} = g_k^*$, for all integer k.

Note

$$u(x,t) = \sum_{m=-\infty}^{m=+\infty} f_m e^{i(mx + m^3 t)}, \quad \frac{\partial^k u}{\partial x^k} = \sum_{m=-\infty}^{m=+\infty} (im)^k f_m e^{i(mx+m^3 t)}.$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_0^{2\pi} = \sum_{m=-\infty}^{m=+\infty} (im)^k f_m e^{im^3 t} \left(e^{2\pi i m} - 1\right) = 0.$$

2. **(Heat equation)** Consider the forced heat equation on the half-line

$$\begin{aligned} u_t = u_{xx} + F(x,t), \quad x \in (0,\infty), \quad t > 0, \\ u(x,0) = 0, \quad x \in (0,\infty). \end{aligned} \tag{1}$$

Prove that $u_D(x,t) \leq u_N(x,t)$ for $x \in (0,\infty)$, $t > 0$ provided $F(x,t) \geq 0$ where $u_D(x,t)$ and $u_N(x,t)$ satisfy (1) subject to homogeneous Dirichlet $u_D(0,t) = 0$ and Neumann $\partial_x u_N(0,t) = 0$ boundary conditions, respectively. *Hint: solve each initial-boundary value problem.*

3. **(Elliptic equation)** Consider the elliptic equation

$$\nabla^2 u(\mathbf{x}) = F(\mathbf{x}), \quad \mathbf{x} \in D \subset \mathbb{R}^n, \tag{2}$$

where D is an open, bounded set.

(a) Suppose

$$\frac{\partial u}{\partial n} + a(\mathbf{x})u = h(\mathbf{x}), \quad \mathbf{x} \in \partial D$$

where h is given on the closed, connected boundary ∂D, $a(\mathbf{x}) > 0$ and $\mathbf{n}$ is the outward unit normal such that $\frac{\partial u}{\partial n} \equiv \boldsymbol{n} \cdot \nabla u$. Utilizing Green's identities, prove that the solution is unique.

(b) Suppose $\frac{\partial u}{\partial n} = g(\mathbf{x})$ for $\mathbf{x} \in \partial D$. Find a necessary condition involving only F, g, and D (not u) for the solution to exist.

(c) Suppose $D = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 2\right\}$, $F = 0$, and $u(\mathbf{x}) = 3\sin 2\theta + 1$ for $\mathbf{x} = (2\cos\theta, 2\sin\theta)$, $\theta \in [0, 2\pi)$. Without solving the equation:

i. Find the maximum value of $u(\mathbf{x})$ for $|\mathbf{x}| \leq 2$.

ii. Find $u(\mathbf{0})$.

4. **(Wave equation)** Consider the Darboux problem

$$u_{tt} = u_{xx}, \quad |x| < t, \quad t > 0,$$
$$u(x,t) = \begin{cases} f(t), & x = t, \quad t \geq 0, \\ g(t), & x = -t, \quad t \geq 0, \end{cases}$$

where $f, g \in C^2([0,\infty))$ satisfy $f(0) = g(0)$.

(a) Solve the Darboux problem. What, if any, are the additional requirements for a classical solution?

(b) Prove that the Darboux problem is well posed.

5. **(Method of characteristics)** Solve the following Cauchy problems and verify your solution.

   (a) $u_y = xuu_x$, $u(x,0) = x$, $x \in \mathbb{R}$.

   (b) $xu_y - yu_x = u$, $u(x,0) = h(x)$, $x \in \mathbb{R}$.