Preliminary Examination (Solutions): Partial Differential Equations
1 PM—4 PM, Jan. 12, 2023,
Newton Lab

#	possible	score
1	25	
2	25	
3	25	
4	25	
5	25	
Total	100	

Student ID:____________________________________

There are five problems. **Solve four of the five problems and circle which four you choose in the grading key on the right.** Each problem is worth 25 points. Please start each problem on a new page. A sheet of convenient formulae is provided.

1. **(Solution methods)** Consider the Airy equation

$$u_t + u_{xxx} = 0, \quad x \in (0, 2\pi), \quad t \in (0, \infty)$$

with periodic boundary conditions $u(0,t) = u(2\pi,t)$, $u_x(0,t) = u_x(2\pi,t)$, and $u_{xx}(0,t) = u_{xx}(2\pi,t)$ for all $t \geq 0$, and initial condition $u(x,0) = f(x)$. The initial condition f is assumed to be real-valued, 2π-periodic, and C^∞.

(a) Show that $M(t) = \int_0^{2\pi} u(x,t)dx$ remains constant in time.

(b) Let $H^k(t) = \int_0^{2\pi} \left[\frac{\partial^k}{\partial x^k} u(x,t)\right]^2 dx$, $k = 0, 1, 2, \ldots$ (H^0 is the L_2 norm of u). Show that H^0 and H^1 remain constant in time. Is this also true for the H^k with higher k?

(c) Give a series solution of the inhomogeneous Airy equation

$$\begin{aligned} u_t + u_{xxx} &= \cos(2x + \omega t), \quad 0 \leq x \leq 2\pi,\ t \geq 0 \\ u(x,0) &= 0, \quad 0 \leq x \leq 2\pi \\ u(0,t) &= u(2\pi,t) \quad t \geq 0 \end{aligned}$$

Discuss the qualitative difference between solutions corresponding to $\omega = 8$ and $\omega \neq 8$.
Useful fact: If $g(x)$ is real and periodic, then its complex Fourier coefficients g_k satisfy $g_{-k} = g_k^*$, for all integer k.

Note

$$u(x,t) = \sum_{m=-\infty}^{m=+\infty} f_m e^{i(mx+m^3t)}, \quad \frac{\partial^k u}{\partial x^k} = \sum_{m=-\infty}^{m=+\infty} (im)^k f_m e^{i(mx+m^3t)}.$$

$$\left.\frac{\partial^k u}{\partial x^k}\right|_0^{2\pi} = \sum_{m=-\infty}^{m=+\infty} (im)^k f_m e^{im^3t}\left(e^{2\pi im} - 1\right) = 0.$$

Solution:

(a)

$$\frac{dM(t)}{dt} = \int_0^{2\pi} u_t dx = -\int_0^{2\pi} u_{xxx} dx = -u_{xx}|_0^{2\pi} = 0$$

(b)

$$\begin{aligned}\frac{dH^k(t)}{dt} = 2\int_0^{2\pi} \frac{\partial^k}{\partial x^k} u \frac{\partial^k}{\partial x^k} u_t dx &= -2\int_0^{2\pi} \frac{\partial^k}{\partial x^k} u \frac{\partial^{k+3}}{\partial x^{k+3}} u dx \\ &= 2\int_0^{2\pi} \frac{\partial^{k+1}}{\partial x^{k+1}} u \frac{\partial^{k+2}}{\partial x^{k+2}} u dx \\ &= \int_0^{2\pi} \frac{\partial}{\partial x} \left(\frac{\partial^{k+1}}{\partial x^{k+1}} u \right)^2 = 0\end{aligned}$$

(c) To obtain a particular solution $p(x,t)$ satisfying the forced equation, note that

$$\cos(2x+\omega t) = \frac{1}{2}\left(e^{i(2x+\omega t)} + e^{-i(2x+\omega t)}\right).$$

Seeking solutions in the form $p(x,t) = p_2(t)e^{2ix} + p_{-2}(t)e^{-2ix}$ that respect periodic boundary conditions,

$$p_2(t) = \frac{i}{2}\frac{e^{i\omega t}}{(8-\omega)}, \quad p_2(t) = u_2^*(t), \quad \text{if } \omega \neq 8.$$

$$p(x,t) = \frac{1}{(\omega-8)}\sin(2x+\omega t),$$

i.e., non-resonant and bounded, or,

$$p_2(t) = \frac{1}{2}te^{8it}, \quad p_2(t) = p_2^*(t), \quad \text{if } \omega = 8.$$

$$p(x,t) = t\cos(2x+8t)$$

resonant and unbounded.

We must also satisfy the initial condition $u(x,0) = 0$. The particular solution for $\omega = 8$ suffices because $p(x,0) = 0$ so the solution to the initial/boundary value problem is $u(x,t) = t\cos(2x+8t)$. This solution is oscillatory and grows with time, i.e., is resonant.

When $\omega \neq 8$, we must add a homogeneous solution $u(x,t) = h(x,t) + \frac{1}{\omega-8}\sin(2x+\omega t)$ such that $h(x,0) = -\frac{1}{\omega-8}\sin(2x)$. The homogeneous solution $h(x,t) = -\frac{1}{\omega-8}\sin(2x+8t)$ does the trick so that the solution to the initial/boundary value problem is $u(x,t) = \frac{1}{\omega-8}\Big(\sin(2x+\omega t) - \sin(2x+8t)\Big)$. This solution is oscillatory and bounded in time.

2. **(Heat equation)** Consider the forced heat equation on the half-line

$$\begin{aligned} u_t = u_{xx} + F(x,t), \quad x \in (0,\infty), \quad t > 0, \\ u(x,0) = 0, \quad x \in (0,\infty). \end{aligned} \tag{1}$$

Prove that $u_D(x,t) \le u_N(x,t)$ for $x \in (0,\infty)$, $t > 0$ provided $F(x,t) \ge 0$ where $u_D(x,t)$ and $u_N(x,t)$ satisfy (1) subject to homogeneous Dirichlet $u_D(0,t) = 0$ and Neumann $\partial_x u_N(0,t) = 0$ boundary conditions, respectively. *Hint: solve each problem.*

Solution: Use Duhamel's principle to recast the forced problem (1) as the family of initial value problems parametrized by $s \ge 0$:

$$\begin{aligned} \tilde{u}_t = \tilde{u}_{xx}, \quad x \in (0,\infty), \quad t > s, \\ \tilde{u}(x, t = s; s) = F(x,s), \quad x \in (0,\infty). \end{aligned} \tag{2}$$

Then the solution to the forced problem is

$$u(x,t) = \int_0^t \tilde{u}(x,t;s)\,\mathrm{d}s.$$

For the Dirichlet problem, perform an odd reflection of the initial data $\tilde{u}(x, t = s; s) = \tilde{F}(x,s)$ where

$$\tilde{F}(x,s) = \begin{cases} F(x,s), & x > 0, \\ -F(-x,s), & x < 0, \end{cases}$$

so that the solution of (2) is

$$\begin{aligned} \tilde{u}_D(x,t;s) &= \int_{-\infty}^{\infty} \Phi(x-y,t-s)\tilde{F}(y,s)\,\mathrm{d}y \\ &= \int_0^{\infty} \Big(\Phi(x-y,t-s) - \Phi(x+y,t-s)\Big)F(y,s)\,\mathrm{d}y, \end{aligned}$$

where $\Phi(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)}$ is the fundamental solution of the heat equation on $\mathbb{R}$. Applying Duhamel's principle, we obtain the solution of the forced Dirichlet problem

$$u_D(x,t) = \int_0^t \int_0^{\infty} \Big(\Phi(x-y,t-s) - \Phi(x+y,t-s)\Big)F(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

The same procedure applies for the Neumann problem except an even reflection of the data is used, which ultimately results in

$$u_N(x,t) = \int_0^t \int_0^{\infty} \Big(\Phi(x-y,t-s) + \Phi(x+y,t-s)\Big)F(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Now directly compute

$$u_N(x,t) - u_D(x,t) = 2\int_0^t \int_0^{\infty} \Phi(x+y,t-s)F(y,s)\,\mathrm{d}y\,\mathrm{d}s \ge 0$$

because $\Phi > 0$ and $F \ge 0$.

3. **(Elliptic equation)** Consider the elliptic equation

$$\nabla^2 u(\mathbf{x}) = F(\mathbf{x}), \quad \mathbf{x} \in D \subset \mathbb{R}^n, \tag{3}$$

where D is an open, bounded set.

(a) Suppose

$$\frac{\partial u}{\partial n} + a(\mathbf{x})u = h(\mathbf{x}), \quad \mathbf{x} \in \partial D$$

where h is given on the closed, connected boundary ∂D, $a(\mathbf{x}) > 0$ and $\mathbf{n}$ is the outward unit normal such that $\frac{\partial u}{\partial n} \equiv \boldsymbol{n} \cdot \nabla u$. Utilizing Green's identities, prove that the solution is unique.

(b) Suppose $\frac{\partial u}{\partial n} = g(\mathbf{x})$ for $\mathbf{x} \in \partial D$. Find a necessary condition involving only F, g, and D (not u) for the solution to exist.

(c) Suppose $D = \left\{\mathbf{x} \in \mathbb{R}^2 \mid |\mathbf{x}| < 2\right\}$, $F = 0$, and $u(\mathbf{x}) = 3\sin 2\theta + 1$ for $\mathbf{x} = (2\cos\theta, 2\sin\theta)$, $\theta \in [0, 2\pi)$. Without solving the equation:

i. Find the maximum value of $u(\mathbf{x})$ for $|\mathbf{x}| \leq 2$.

ii. Find $u(\mathbf{0})$.

Solution:

(a) Let u_1 & u_2 be solutions to (1). Consider $w = u_1 - u_2$ such that w satisfies

$$\nabla^2 w = 0, \quad \mathbf{x} \in D, \quad \text{and} \quad \frac{\partial w}{\partial n} + a(\mathbf{x})w = 0, \quad \mathbf{x} \in \partial D.$$

Multiply the boundary constraint by w, integrate over ∂D and utilize Green's theorem

$$\begin{aligned} -\int_{\partial D} a(\mathbf{y}) w^2(\mathbf{y})\,\mathrm{d}S_y &= \int_{\partial D} w \frac{\partial w}{\partial n}\,\mathrm{d}S_y \\ &\quad \text{by Green's Thm} \\ &= \int_D \left(w\nabla^2 w + \nabla w \cdot \nabla w\right)\mathrm{d}\mathbf{x} \\ &= \int_D |\nabla w|^2\,\mathrm{d}\mathbf{x} \\ &= 0 \quad \text{because both integrands are nonnegative definite.} \end{aligned}$$

Then $\nabla w \equiv 0$ in D implying $w =$const. in $\overline{D}$. Furthermore, $\frac{\partial w}{\partial n} = 0$ on ∂D. Thus, $w \equiv 0$ by the imposed boundary constraint. Hence $u_1 \equiv u_2$ and the solution is unique.

(b) Suppose $\frac{\partial u}{\partial n} = g(\mathbf{x})$ on ∂D, by Green's Identity

$$\int_D \nabla^2 u\,\mathrm{d}\mathbf{x} = \int_{\partial D} \frac{\partial u}{\partial n}\,\mathrm{d}S_y \quad \Longrightarrow \quad \int_D F(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\partial D} g(\mathbf{y})\,\mathrm{d}S_y$$

(c) i. By the maximum principle, the maximum must occur on the boundary which has a max value of 4.

ii. The center $u(\mathbf{0})$ is equal to the average of u on the boundary:

$$\begin{aligned}\overline{u(2,\theta)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(3\sin 2\theta + 1)\,d\theta \\ &= \left.\frac{-\frac{3}{2}\cos 2\theta + \theta}{2\pi}\right|_{-\pi}^{\pi} = 1.\end{aligned}$$

4. **(Wave equation)** Consider the Darboux problem

$$u_{tt} = u_{xx}, \quad |x| < t, \quad t > 0,$$

$$u(x,t) = \begin{cases} f(t), & x = t, \quad t \geq 0, \\ g(t), & x = -t, \quad t \geq 0, \end{cases}$$

where $f, g \in C^2([0,\infty))$ satisfy $f(0) = g(0)$.

(a) Solve the Darboux problem. What, if any, are the additional requirements for a classical solution?

(b) Prove that the Darboux problem is well posed.

Solution:

(a) The general solution of the wave equation is $u(x,t) = F(x+t) + G(x-t)$. Applying the boundary conditions

$$u(t,t) = F(2t) + G(0) = f(t) \quad \Rightarrow \quad F(z) = f(z/2) - G(0),$$
$$u(-t,t) = F(0) + G(-2t) = g(t) \quad \Rightarrow \quad G(z) = g(-z/2) - F(0).$$

Since $f(0) = F(0) + G(0) = g(0)$, the solution is

$$u(x,t) = f\left(\frac{x+t}{2}\right) + g\left(\frac{t-x}{2}\right) - f(0).$$

Since $\lim_{t\to 0} u(t,t) = g(0)$, $\lim_{t\to 0} u(-t,t) = g(0)$, the solution is continuous up to and including the origin. Since $f, g \in C^2([0,\infty))$, $u_{tt}(x,t)$ and $u_{xx}(x,t)$ are continuous up to and including the boundaries so the solution is a classical solution.

(b) We have already proven existence of a solution. Uniqueness is due to the unique determination of F and G from the boundary data for the general solution of the wave equation. For continuous dependence, consider two solutions $u_1(x,t)$ and $u_2(x,t)$ of two Darboux problems subject to boundary data $(f_1(t), g_1(t))$ and $(f_2(t), g_2(t))$, respectively. Suppose that

$$|f_1(t) - f_2(t)| < \delta, \quad |g_1(t) - g_2(t)| < \delta,$$

for all $t \geq 0$, then for all $|x| \leq t$ and $t \geq 0$, we have

$$\begin{aligned} |u_1(x,t) - u_2(x,t)| &\leq \left| f_1\left(\frac{x+t}{2}\right) - f_2\left(\frac{x+t}{2}\right)\right| + \left| g_1\left(\frac{t-x}{2}\right) - g_2\left(\frac{t-x}{2}\right)\right| \\ &\quad + |f_1(0) - f_2(0)| \\ &< \delta + \delta + \delta = 3\delta. \end{aligned}$$

Therefore, for any given $\epsilon > 0$, we set $\delta < \epsilon/3$ so that for all $|x| \leq t$ and $t \geq 0$, we have

$$|u_1(x,t) - u_2(x,t)| < \epsilon,$$

i.e., if the boundary data are close, so are the solutions.

5. **(Method of characteristics)** Solve the following Cauchy problems and verify your solution.

   (a) $u_y = xuu_x$, $u(x,0) = x$, $x \in \mathbb{R}$.

   (b) $xu_y - yu_x = u$, $u(x,0) = h(x)$, $x \in \mathbb{R}$.

Solution:

(a) $\mathrm{d}u/\mathrm{d}y = 0$ along $\mathrm{d}x/\mathrm{d}y = xu$. Then

$$u(x,y) = x_0, \quad \ln|x| = -x_0 y + \ln|x_0|.$$

Exponentiating, we obtain the implicit solution

$$x = ue^{-uy}.$$

Implicit partial differentiation with respect to y yields

$$0 = u_y e^{-uy} - ue^{-uy}(u_y y + u) \quad \Rightarrow \quad u_y = \frac{u^2}{1-yu}.$$

Implicit partial differentiation with respect to x yields

$$1 = u_x e^{-uy} + ue^{-uy}(-u_x y) \quad \Rightarrow \quad u_x = \frac{e^{uy}}{1-yu} = \frac{u}{x(1-yu)}.$$

Then, $u_y - xuu_x = 0$ and $x = u(x,0)e^{-u(x,0)0} = u(x,0)$ as required.

(b) Rewrite the dependent variable in polar form $u = u(r,\theta)$ where $x = r\cos\theta$, $y = r\sin\theta$ and $r = \sqrt{x^2+y^2}$, $\theta = \arctan(y/x)$. Then,

$$u_\theta = u_x x_\theta + u_y y_\theta = -u_x y + u_y x = u.$$

The solution is

$$u(r,\theta) = f(r)e^\theta \quad \Rightarrow \quad u(x,y) = f\left(\sqrt{x^2+y^2}\right)e^{\arctan(y/x)}.$$

Applying the Cauchy data at $y = 0$,

$$u(x,0) = f(|x|) = h(x) \quad \Rightarrow \quad u(x,y) = h\left(\sqrt{x^2+y^2}\right)e^{\arctan(y/x)}.$$

To verify the solution, compute

$$u_y = \frac{yh'\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}e^{\arctan(y/x)} + h\left(\sqrt{x^2+y^2}\right)e^{\arctan(y/x)}\frac{x}{x^2+y^2},$$
$$u_x = \frac{xh'\left(\sqrt{x^2+y^2}\right)}{\sqrt{x^2+y^2}}e^{\arctan(y/x)} - h\left(\sqrt{x^2+y^2}\right)e^{\arctan(y/x)}\frac{y}{x^2+y^2}.$$

Then, $xu_y - yu_x = u$ and $u(x,0) = h(x)$ as required.