Chapter 3

Analytic continuation

A particular representation of a function might be valid only in a limited region of the complex plane, although the function itself may exist well past that region. For example, a function defined through its Taylor series can only be evaluated (by immediate summation) inside this expansion’s circular domain of convergence, which is limited in size by the distance to the nearest singularity.

3.1 Introductory examples

Example 3.1.1 Consider the following three functions

1. \[ f_1(z) = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \ldots \] From Theorem 2.3.1 (Parts 2, 3, or 4) follows that the radius of convergence \( R = 1 \).

2. \[ f_2(z) = \int_0^{\infty} e^{-t(1-z)} dt \] If the integral had been over a finite interval rather than over \([0, \infty]\), it would have defined an entire function. With the upper limit \( \infty \), it will converge for Re \( z < 1 \) but diverge for Re \( z \geq 1 \).

3. \[ f_3(z) = \frac{1}{1-z} \] This expression represents an analytic function in the whole complex plane, with the exception of a single first order pole located \( z = 1 \).

The key point in this example is that all the three functions are identically the same. The boundaries for \( f_1(z) \) and \( f_2(z) \) are not natural ones, but just artifacts of the particular way we used for representing the function \( f_3(z) \).

The goal of analytic continuation is to reformulate functional expressions so they can be used in larger regions. It can occur that continuation is not possible. In the following example, the unit circle forms a natural boundary, and it is not possible to continue the function past it.

Example 3.1.2 Consider the function \( f(z) = \sum_{k=1}^{\infty} z^k = z + z^2 + z^6 + z^{24} + \ldots \) Part 3 of Theorem 2.3.1 gives again \( R = 1 \), and the domain of convergence will be the same as for the function \( f_1(z) \) in Example 3.1.1. Consider a point \( z = e^{2\pi ik/m} \) where \( k \) and \( m \) are arbitrary integers. Then \( z^n = e^{2\pi ink/m} = 1 \) for all \( n \geq m \). This means that \( f(z) \to +\infty \) whenever \( z \) is of the form \( z = re^{2ik/m} \) and \( r \to 1 \). The function \( f(z) \) must therefore have singularities located densely all the way around the unit circle, making \( |z| = 1 \) a natural boundary. It can not be continued past it.

What does a function like this look like graphically? Figures 3.1 (a,b) shows its magnitude and phase angles when displayed within circles of radii 0.99 and 0.999999, respectively. There is of course no difference between the two displays throughout the first region, but we see more fine structure entering near the edge in the second case. Moving even closer to the unit circle, the fine structure becomes infinitely dense. We can also note that \(|f(z)|\) does not increase monotonically as the unit circle is approached - indeed, we can in the phase angle color patterns see tell-tale signs of increasing numbers of zeros. Near the origin, this function...
(a) Display of $|f(z)|$ over $|z| \leq 1 - 10^{-2}$.

(b) Display of $|f(z)|$ over $|z| \leq 1 - 10^{-6}$.

Figure 3.1: Illustration of $|f(z)|$ (with phase coloring) of $f(z) = \sum_{k=1}^{\infty} z^k$ over two slightly different circles around the origin. Illustration of a natural boundary.

$f(x)$ becomes of course very close to $f(z) = z$, as displayed in the top right subplot of Figure 2.1. □

Whenever a function is given in a form which only permits it to be evaluated in some limited region of the complex plane, it becomes of great interest to see if it can be rewritten if some other form that permits the domain to be extended - preferably to the complete complex plane. It is quite common in this context that the initial representation is a Taylor series.

3.2 Some methods for analytic continuation

Each of the following nine subsections describes a different method for analytic continuation:

1. Circle-chain method
2. Schwarz reflection
3. Use of a functional equation
4. Partitioning of an integration interval
5. Replace Taylor coefficients by integrals or sums
6. Subtract similar series / integral
7. Borel summation
8. Padé approximations
9. Ramanujan’s formula

Especially the Padé approach is noteworthy also from the point of view that it can provide quite accurate continuations already from highly incomplete information, such as from a relatively low number of leading terms in truncated Taylor expansions. Still further examples of continuation methods (that require the use of complex integration) are given in Section 4.6.
3.2. SOME METHODS FOR ANALYTIC CONTINUATION

3.2.1 Circle-Chain method

This approach is seldom practical for actual applications, but it is of interest in that it gives insights into what analytic continuation amounts to. Suppose that we have a function with point-type singularities (poles, branch points or essential singularities) in the locations marked by black dots in Figure 3.2. If we are given initially a Taylor expansion centered at the origin of the complex plane, it will converge only in the domain bounded by the solid circle, extending out to the nearest singularity. Within this circle, the expansion completely describes the function. Hence, we can choose any other point inside it and create a new expansion centered at this new location. Its domain of convergence will likely reach outside the original one. The process can in theory be repeated indefinitely, allowing us to proceed for example as is indicated by the chain of circles in this figure.

From a practical (as opposed to a theoretical) perspective, this approach is seldom useable. If a complete expansion is known analytically, it is rare that one can find convenient closed forms also for the subsequent expansions (however one such example is given as Example 3.2.1 and another one as Exercise 3.3.1). If the initial expansion is only known numerically, this continuation process becomes extremely ill conditioned.

This circle-chain method also requires the initial Taylor expansion’s radius of convergence $R$ to be non-zero. Although the theory for continuation based on an $R = 0$ initial expansion is unclear, at least three of the following approaches can nevertheless do this successfully - see Example 3.2.12 and Exercises 3.3.3 (b,c) below.

An interesting situation arises if one tries to continue all the way around a singularity. Will the results agree when the circle-chain continuation comes back to the start point? The answer will be ‘Yes’ when going around poles and essential singularities, but ‘No’ in the case of a branch point.

Example 3.2.1 Continue the function $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$ by means of a chain of circles once in the clockwise direction around the singularity at $z = 1$.

We can start by noting (but will not use further) the fact that $f(z) = -\log(1 - z)$, so the end result after having step-by-step continued around the singularity should return the same function as we started with, apart from an increase in its imaginary part by $2\pi i$.

The original expansion is centered at $z_0 = 0$. Let the subsequent expansions be centered at $z_1, z_2, \ldots, z_p = z_0$ where $z_k = 1 - e^{-2\pi k/p}$, $k = 1, 2, \ldots, p$ (see Figure 3.3). From the Taylor expansion of $f(z)$ follows

$$f'(z) = \sum_{n=1}^{\infty} z^{n-1} = \frac{1}{1 - z} \quad (3.1)$$
and therefore \( f^{(m)}(z) = \frac{(m-1)!}{(1-z)^m} \). The Taylor expansion centered at \( z_1 \) therefore becomes

\[
f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_1)(z - z_1)^n = f(z_1) + \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{z - z_1}{1 - z_1} \right)^n = f(z_1) + f \left( \frac{z - z_1}{1 - z_1} \right)
\]  

(3.2)

Differentiating (3.2), we obtain from its first and third expressions

\[
f'(z) = \frac{1}{1 - z_1} \sum_{n=1}^{\infty} \left( \frac{z - z_1}{1 - z_1} \right)^{n-1} = \frac{1}{1 - z},
\]

confirming that (3.1) again holds when now expanding around \( z = z_1 \) (and its different domain of convergence). Looking at the first and last expressions in (3.2), a pattern becomes clear. For the next expansion, around the point \( z_2 \), we get similarly

\[
f(z) = f(z_1) + f \left( \frac{z_2 - z_1}{1 - z_1} \right) + f \left( \frac{z - z_2}{1 - z_2} \right),
\]

etc. After \( p \) steps, we arrive at

\[
\begin{align*}
\text{new } f(z) = f(z_1) &+ f \left( \frac{z_2 - z_1}{1 - z_1} \right) + f \left( \frac{z_3 - z_2}{1 - z_2} \right) + \ldots + f \left( \frac{z_p - z_{p-1}}{1 - z_{p-1}} \right) + \frac{z - z_p}{1 - z_p} \\
\text{additive constant} &+ f \left( \frac{z - z_1}{1 - z_1} \right)
\end{align*}
\]

(3.3)

Are we back? Noting that \( z_k = 1 - e^{-2 \pi i k/p} \), all the \( p \) terms in the ‘additive constant’ become equal to \( f(z_1) \) with \( z_1 = 1 - e^{-2 \pi i/p} \). Furthermore, the value of this additive constant must be a fixed number, independent of \( p \). To determine its value, we can let \( p \to \infty \). This gives

\[
p f(z_1) = p \left( 1 - e^{-2 \pi i/p} \right) \frac{f(z_1)}{z_1} = p \left( 1 - e^{-2 \pi i/p} \right) \left( \frac{z_1}{2} + \frac{z_1^2}{3} + \ldots \right),
\]

which shows that it must evaluate to \( 2 \pi i \). Continuation with the circle-chain method all the way around the singularity at \( z = 1 \) thus did not come back to the same expansion as we started with, telling that the singularity was a branch point. Every time we continue in the clockwise direction around \( z = 1 \), this function \( f(z) \) increases in value by \( 2 \pi i \). □.

Figure 3.3: Illustration of the sequence of Taylor expansion centers in Example 3.2.1.
3.2. SOME METHODS FOR ANALYTIC CONTINUATION

3.2.2 Schwarz reflection principle

Suppose that we are given a function \( f(z) \) that is

1. Known to be analytic in some region \( D \) (cf. Figure 3.4), and
2. Real on the real axis (or on some part of it)

We showed in Section 2.1.2, last bullet item, that the function \( g(z) = \overline{f(\bar{z})} \) will then also be analytic. However, this \( g(z) \) is not defined on \( D \) but on \( \overline{D} \) (the reflection of \( D \) in the real axis). Since \( g(z) \) agrees with \( f(z) \) along a section of the real axis, the two functions must be the same (Theorem 2.3.2). The function as \( f(z) \) has thus been continued from \( D \) to \( \overline{D} \). Another way to formulate the result is:

**Theorem 3.2.1** An analytic function \( f(z) \) that is real on any segment of the real axis takes complex conjugated values at complex conjugated locations in the \( z \)-plane.

The result can be generalized significantly. For any analytic function, there will be some curve(s) along which \( \text{Im} f(z) = 0 \). If we can find some analytic change of variable \( z = z(w) \) such that the segment corresponds to real values of \( w \), we can apply the reflection principle to \( f(z(w)) \).

![Figure 3.4: Schematic illustration of the Schwarz reflection principle.](image)

3.2.3 Use of a functional equation

We will illustrate this approach by three examples. The first two extend \( \Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \), convergent for \( \text{Re} \ z > 0 \), to the whole complex plane.

**Example 3.2.2** Continue the Gamma function by utilizing the functional equation

\[
\Gamma(z + 1) = z \Gamma(z)。
\]

We noted in Section 2.7 that the functional equation follows from integration by parts in the defining integral, and is valid (like the integral for \( \Gamma(z) \)) throughout \( \text{Re} \ z > 0 \). If we introduce \( H(z) = \frac{\Gamma(z+1)}{z} \), it will thus hold that \( H(z) = \Gamma(z) \) when \( \text{Re} \ z > 0 \). Recalling Theorem 2.3.2, the functions \( H(z) \) and \( \Gamma(z) \) are identical.

However, with \( \Gamma(z) \) defined for \( \text{Re} \ z > 0 \), \( H(z) = \frac{\Gamma(z+1)}{z} \) becomes defined for \( \text{Re} \ z > -1 \), and thus, so is now \( \Gamma(z) \). This has continued \( \Gamma(z) \) through the strip \( -1 < \text{Re} \ z \leq 0 \) (and we can note that \( \Gamma(z) \) will have a pole at \( z = 0 \)). This process can be repeated indefinitely, showing that \( \Gamma(z) \) exists uniquely defined throughout the entire complex plane, and that it has poles at \( z = 0, -1, -2, ... \) as its only singularities. □
Example 3.2.3  Continue the Gamma function by utilizing the functional equation
\[ \Gamma(z)\Gamma(1 - z) = \frac{\pi}{\sin \pi z}. \]  (3.3)

Two different proofs for this equation will be given later (Theorem 4.4.2 and in the comments following Theorem 5.1.2; see also Example 3.2.14). This relation gives the value of the \( \Gamma \)-function at the location \( z \) whenever it is available at a location \( 1 - z \). Whenever \( \text{Re } z < 0 \) (i.e. we are outside where the integral definition holds), \( \text{Re } (1 - z) \geq 1 \), so \( \Gamma(1 - z) \) can be evaluated. In this case, \( \Gamma(z) \) has in a single step been continued over the whole complex plane (omitting singularities). □

The next example requires a convergence theorem for Dirichlet series:

**Theorem 3.2.2**  The regions of convergence and divergence for a Dirichlet series \( \sum_{n=1}^{\infty} \frac{a_n}{n^z} \) are half-planes \( \text{Re } z > \alpha \) and \( \text{Re } z < \alpha \) respectively, with \( \alpha \) the least value that makes the right half-plane singularity free.

**Example 3.2.4**  Continue the Riemann zeta function
\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \]  (4.3)

with use of the functional equation
\[ \zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{z \pi}{2}\right) \Gamma(1 - z) \zeta(1 - z) \]  (3.5)

The functional equation (cf. Theorem 5.3.2) will immediately (just as was the case for the Gamma function in the above example) give the continuation everywhere that \( \text{Re } z \leq \frac{1}{2} \) if values are available for \( \text{Re } z \geq \frac{1}{2} \).

By Theorem 3.2.2 above, we can find the convergence boundary by inspecting the sum (4.3) for \( z = x \) real. Comparing the sum \( \sum_{n=1}^{\infty} \frac{1}{n^z} \) to the integral \( \int_{1}^{\infty} \frac{dt}{t^z} = \frac{1}{z-1} \) shows that (4.3) converges for \( \text{Re } z > 1 \). What remains for us is therefore to somehow continue (4.3) from \( \text{Re } z > 1 \) down to \( \text{Re } z \geq 1/2 \), after which (3.5) completes the task of continuation to the whole complex plane.

One idea for this is to turn the sum in (4.3) into an alternating one. From
\[ \zeta(z) = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \frac{1}{5^z} + \frac{1}{6^z} + \ldots \]
follows
\[ \frac{2}{2^z} \zeta(z) = \frac{2}{2^z} + \frac{2}{4^z} + \frac{2}{6^z} + \ldots \]
Subtracting this second series from the first one gives
\[ \left(1 - 2^{1-z}\right) \zeta(z) = \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + \ldots \]  (3.6)

This new sum is again of Dirichlet type, and it therefore suffices to test convergence for \( z = x \) real. Since the sum in this case alternates and the magnitudes of the terms decay for all \( x > 0 \), it will, by Theorem 3.2.2, also converge for all \( z \) with \( \text{Re } z > 0 \). The continuation of 3.4 is complete. □

As a sideline, we can note that
\[ \lim_{z \rightarrow 1} \frac{1 - 2^{1-z}}{1 - z} = \lim_{z \rightarrow 1} \frac{1 - e^{(1-z) \log 2}}{1 - z} = - \log 2 \]
and
\[ \lim_{z \rightarrow 1} \left( \frac{1}{1^z} - \frac{1}{2^z} + \frac{1}{3^z} - \frac{1}{4^z} + \frac{1}{5^z} - \frac{1}{6^z} + \ldots \right) = \log 2, \]
from which (3.6) gives that \( \lim_{z \rightarrow 1}(z - 1)\zeta(z) = 1 \). One can then deduce from (3.5) that the pole of \( \zeta(z) \) at \( z = 1 \) is the only singularity of the \( \zeta(z) \)-function in the complex plane (and also that \( \zeta(0) = -\frac{1}{12} \)). We can further note a few other values of the zeta function. Evaluating \( \zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} \) is known as the Basel problem, shown by Euler in 1734 to evaluate to \( \frac{\pi^2}{6} \). Using contour integration, we will obtain this result as a special case of evaluating \( \zeta(2m) \), \( m = 1, 2, 3, \ldots \) in Example 4.5.2. With this value for \( \zeta(2) \), the functional equation gives \( \zeta(-1) = -\frac{1}{12} \). Finding similar closed form expressions for \( \zeta(k) \) when \( k = 3, 5, 7, \ldots \) remains an unsolved problem.
3.2.4 Partitioning of an integration interval

**Example 3.2.5** Extend \( \Gamma(z) \) (again) from its integral definition (2.17), but without use of a functional relation.

In the integral definition \( \Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt \), the upper integration limit causes no problem (thanks to the fast decay of the exponential function). The limitation \( \text{Re} \, z > 0 \) comes from what happens at the origin. Thus, we split the interval in two parts:

\[
\Gamma(z) = \int_0^1 e^{-t}t^{z-1}dt + \int_1^\infty e^{-t}t^{z-1}dt
\]

Analytic for \( \text{Re} \, z > 0 \); Continue this part Analytic for all \( z \); Entire function.

On the short interval \([0, 1]\), the Taylor expansion of \( e^{-t} \) converges rapidly, giving

\[
\int_0^1 t^{z-1}e^{-t}dt = \int_0^1 t^{z-1} \left( \sum_{n=0}^\infty \frac{(-t)^n}{n!} \right) dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{z-1+n}dt
\]

\[= \frac{1}{z+n} \quad \text{if Re} \, z > -n\]

So it might not look like we have achieved anything extra. The first term in the sum has \( n = 0 \), so the restriction would still seem to be \( \text{Re} \, z > 0 \). However, let us ignore this and consider the function

\[
\eta(z) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty e^{-t}t^{z-1}dt
\]

Converges for all \( z \). Poles at \( z = 0, -1, -2, \ldots \).

Since \( \eta(z) = \Gamma(z) \) for \( \text{Re} \, z > 0 \), it becomes (again by Theorem 2.3.2) the continuation of \( \Gamma(z) \) to the whole complex plane. \( \square \)

3.2.5 Replace Taylor coefficients by integrals or sums

Again, we describe the approach by means of an example:

**Example 3.2.6** Continue the function

\[
f(z) = \sum_{n=1}^\infty \frac{z^n}{\sqrt{n}}.
\]

The Taylor series converges for \( |z| < 1 \) and diverges for \( |z| > 1 \). We start by noting that a simple change of variable in the relation \( \frac{\sqrt{z}}{2} = \int_0^\infty e^{-x^2}dx \) gives \( \frac{1}{\sqrt{n}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-nx^2}dx \). Therefore

\[
f(z) = \frac{2}{\sqrt{\pi}} \sum_{n=1}^\infty z^n \int_0^\infty e^{-nx^2}dx = \frac{2}{\sqrt{\pi}} \int_0^\infty \left( \sum_{n=1}^\infty z^ne^{-nx^2} \right) dx = \frac{2z}{\sqrt{\pi}} \int_0^\infty \frac{dx}{e^{x^2} - z}.
\]

In the last equality, we made use of the fact that the sum inside the integral is a geometric series and therefore can easily be summed in closed form. The function \( f(z) \) is now uniquely defined everywhere away from a branch cut along the line \( z \geq 1 \). \( \square \)

3.2.6 Subtraction of a similar series or integral

**Example 3.2.7** Continue the function \( f(z) = \sum_{k=0}^\infty \frac{z^k}{1\pm z^2} \).

The original series diverges for \( |z| > 1 \). For large \( k \), the terms become very close to those of the series \( \sum_{k=0}^\infty z^k = \frac{1}{1-z} \). This makes it natural to look at the difference between these two sums. After minor simplifications:

\[
f(z) - \frac{1}{1-z} = - \sum_{k=0}^\infty \frac{(z/2)^k}{1 + 2^{-k}} = -f(z/2).
\]
Hence, we stumbled on to a functional equation

\[ f(z) - \frac{1}{1 - z} = -f(z/2). \]  

\[ (3.9) \]

Since the original sum works for \(|z| < 1\), (3.9) allows us to compute \(f(z)\) for \(|z| < 2\); then repeating again, for \(|z| < 4\), etc. The function is thereby continued to the whole complex plane. It will have poles at \(z = 1, 2, 4, 8, \ldots\)

It is tempting to write (3.9) as \(f(z) = \frac{1}{1 - z} - f(z/2)\) and then apply this relation repeatedly, to obtain \(f(z) = \frac{1}{1 - z} - \frac{1}{1 - z/2} + \frac{1}{1 - z/4} - \ldots\). However, this sum diverges for all \(z\). A much better idea is to repeat the original subtraction idea, first on the sum in (3.8) and then on the sum this gives rise to, etc. After some manipulations, this gives

\[ f(z) = \frac{1}{2} + z \left( \frac{1}{1 - z} - \frac{1/2}{1 - z/2} + \frac{1/4}{1 - z/4} - \frac{1/8}{1 - z/8} + \ldots \right), \]

which converges rapidly for all values of \(z\). □

### 3.2.7 Borel summation

Assume that we are given a function \(f(z)\) defined by a Taylor series \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) that it is convergent in \(|z| < R\). Then

\[ \phi(z) = \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} \]

is an entire function. Next, we form the function

\[ \eta(z) = \int_0^{\infty} e^{-t} \phi(zt) dt = \int_0^{\infty} e^{-t} \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} dt = \sum_{k=0}^{\infty} \frac{a_k z^k}{k!} \int_0^{\infty} e^{-t} \frac{dt}{k!} = \sum_{k=0}^{\infty} a_k z^k = f(z) \]

This new function \(\eta(z)\) clearly agrees with \(f(z)\) within the domain of convergence \(|z| < R\). But is it a continuation of \(f(z)\)? It can be shown that \(\eta(z)\) will converge in the smallest polygon that can be constructed from the singularities of \(f(z)\) in the way that is indicated in Figure 3.5. In this schematic figure, the solid dots mark the singularities of \(f(z)\) and the circle shows where the Taylor expansion around the origin will converge. Through each singularity, we draw a line orthogonal to the direction to the origin. The function \(\eta(z)\) will converge inside the resulting polygon (marked by solid lines in the figure).

**Example 3.2.8** Apply Borel summation to \(f(z) = \sum_{k=0}^{\infty} z^k = 1 + z + z^2 + z^3 + \ldots\)

We get immediately \(\phi(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z\), and therefore the Borel extension takes the form \(f(z) = \int_0^{\infty} e^{-t} \phi(zt) dt = \int_0^{\infty} e^{-t} (1 - z) dt\). This has brought us from \(f_1(z)\) to \(f_2(z)\) to in Example 3.1.1. As we noted there, the new form converges for \(\Re z < 1\), entirely agreeing with the description above of where integrals obtained through Borel summation will converge. □

### 3.2.8 Padé approximations

This methodology can be extremely powerful, especially if all that is known of a Taylor expansion is the numerical values of a finite number of leading coefficients (insufficient information for most other analytic continuation methods).

Taylor expansions (and often also asymptotic expansions, cf. Chapter 10) can often be accelerated quite dramatically (or turned from divergent to convergent) by being rearranged into a ratio of two such expansions.

A Padé approximation

\[ P_M^N(z) = \frac{\sum_{n=0}^{M} a_n z^n}{\sum_{n=0}^{N} b_n z^n} \]  

\[ (3.11) \]
3.2. SOME METHODS FOR ANALYTIC CONTINUATION

Figure 3.5: Schematic illustration of the region of convergence of a Borel sum.

(normalized by \( b_0 = 1 \)) generalizes the Taylor expansion with equally many degrees of freedom

\[
T_{M+N}(z) = \sum_{n=0}^{M+N} c_n z^n, \tag{3.12}
\]

with the two being the same in case \( M = 0 \). The Padé coefficients are normally found by starting from a Taylor expansion:

\[
c_0 + c_1 z + c_2 z^2 + ... = \frac{a_0 + a_1 z + a_2 z^2 + ...}{1 + b_1 z + b_2 z^2 + ...}
\]

and then require that both sides match to the highest degree possible at the origin. Multiplying up the denominator gives the following equivalent set of coefficient relations

\[
\begin{align*}
a_0 &= c_0 \\
a_1 &= c_1 + c_0 b_1 \\
a_2 &= c_2 + c_1 b_1 + c_0 b_2 \\
a_3 &= c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3 \\
&... 
\end{align*} \tag{3.13}
\]

With the \( c_i \) given, each new line introduces two new unknowns, \( a_i \) and \( b_i \). The system would appear to be severely under-determined. However, if we specify the degree of the numerator to be \( N \), of the denominator to be \( M \), and of the truncated Taylor expansion to be \( M + N \), there will be just as many equations as unknowns (ignoring all terms that are \( O(z^{M+N+1}) \)). We can then solve for all the unknown coefficients, as the following example shows:

**Example 3.2.9** Given \( T_5(z) \), determine \( P_3^2(z) \).

In this case of \( M = 3, N = 2, M + N = 5 \), the system (3.13) becomes cut off as follows:

\[
\begin{align*}
a_0 &= c_0 \\
a_1 &= c_1 + c_0 b_1 \\
a_2 &= c_2 + c_1 b_1 + c_0 b_2 \\
a_3 &= c_3 + c_2 b_1 + c_1 b_2 + c_0 b_3 \\
&... 
\end{align*}
\]

...
In the simplest (which is not the most robust) version of the Padé method, we first solve the bottom three equations for $b_1, b_2, b_3$, after which the top three explicitly give $a_1, a_2, a_3$. This same idea carries through for any values of $M$ and $N$. □.

A key usage of Padé approximations is to extract the information from power series expansions with only a few known terms. Transformation to Padé form usually accelerates convergence, and often allows good approximations to be found even well outside a Taylor expansion’s radius of convergence (which might even be $R = 0$).

**Example 3.2.10** Find the increasing order Padé approximations for $f(z) = 1 - z + z^2 - z^3 + -....$

The Padé table based on the truncated Taylor sums becomes as shown in Table 3.1. The main diagonal (and the diagonal below it) usually gives the best results. This example is trivial (and atypical) in that every entry with $M > 0$ happens to recover the exact result. □

<table>
<thead>
<tr>
<th>$N$ - order of numerator</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$ - order of denominator</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>$1 - z$</td>
<td>$1 - z + z^2$</td>
<td>$1 - z + z^2 - z^3$</td>
<td>...</td>
</tr>
<tr>
<td>1</td>
<td>$1/z$</td>
<td>$1/z$</td>
<td>$1 + z$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$1/z^2$</td>
<td>$1/z$</td>
<td>$1 + z$</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$1/z^3$</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

**Example 3.2.11** Approximate $f(2)$ when we only know the first few terms in the Taylor expansion $f(z) = 1 - \frac{1}{2}z + \frac{1}{3}z^2 - \frac{1}{4}z^3 + \frac{1}{5}z^4 - +.... \left(= \frac{\ln(1+z)}{z}, \text{but convergent only if } |z| < 1 \right)$.

The Padé table 3.2 is laid out like Table 3.1, but it shows only the numerical values for $z = 2$ and, in parenthesis, the errors in these compared to $\frac{1}{2} \log 3 \approx 0.5493$. In spite of $z = 2$ being well outside the domain of convergence for the Taylor series for $f(z)$ (as is again visible from the top $M = 0$ row in the table), the Padé method allows fast and accurate calculation of the analytically continued result at $z = 2$.

**Example 3.2.12** Compare Taylor- and Padé approximations for the Stieltjes’ function $f(z) = \int_0^\infty \frac{e^{-t}}{1+z t} \, dt$.

The integral defining $f(z)$ is singular for $z < 0$ (real and negative), but it is well defined for other values of $z$ in the complex plane. Figures 3.6 (a,b) show the result of a direct evaluation of $f(z)$. The presence of a branch discontinuity along the negative real axis is obvious.

We next Taylor expand $f(z)$ around $z = 0$ (for example by repeated integration by parts, or by noting that $f(z)$ satisfies $z^2 f'(z) + (1 + z) f(z) - 1 = 0$, $f(0) = 1$, and then equate coefficients). The resulting expansion becomes

$$f(z) \approx \sum_{k=0}^{\infty} (-z)^k k! .$$

This diverges for all values of $z \neq 0$ (its radius of convergence is $R = 0$). Truncation after the sixth power gives

$$T_6(z) \approx 1 - z + 2z^2 - 6z^3 + 24z^4 - 120z^5 + 720z^6 .$$
**Table 3.2**: Values for \( f(2) \) from Padé approximations; in parentheses their difference to \( \frac{1}{2} \log 3 \).

<table>
<thead>
<tr>
<th>( M )-order of denominator</th>
<th>( N )-order of numerator</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1.3333</td>
<td>-0.6667</td>
<td>2.5333</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.4507)</td>
<td></td>
<td>(−0.5493)</td>
<td>(0.7840)</td>
<td>(−1.2160)</td>
<td>(1.9840)</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.5714</td>
<td></td>
<td>0.5507</td>
<td>(0.0014)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td>0.5494</td>
<td>(0.0001)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td>0.5493</td>
<td>(0.0000)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td>0.5493</td>
<td>(0.0000)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| ...                          |                             |   | ... | ... | ...

(a) Real part of \( f(z) = \int_0^\infty \frac{e^{-t}}{1+zt} dt \).

(b) Imaginary part of \( f(z) = \int_0^\infty \frac{ze^{-t}}{1+zt} dt \).

Figure 3.6: The real and imaginary parts of the Stieltjes’ function.

As seen in Figure 3.7 (a), \( T_6(z) \) resembles the true Stieltjes function at most in a very small region near to the origin. However, after converting \( T_6(z) \) to its \( P_3^3(z) \) Padé counterpart

\[
P_3^3(z) = \frac{1 + 11z + 26z^2 + 6z^3}{1 + 12z + 36z^2 + 24z^3},
\]

we see in Figure 3.7 (b) a somewhat respectable approximation of the original function. Being a rational function, and therefore single-valued, the Padé approximant cannot reproduce the branch discontinuity. It has however placed its pole singularities along the negative real axis in an attempt to mimic the branch discontinuity there (maybe a bit surprising, since the location of a branch cut is not uniquely determined). Finally, Figures 3.7 (c,d) compare \( T_{30}(z) \) with \( P_{15}^3(z) \). In the case of the Stieltjes’ function, it can be proven that the Padé approximations will converge to the true function exponentially fast (as higher degrees are used) everywhere in the complex plane away from the negative real axis. Somehow, the everywhere divergent Taylor expansion \( (R = 0) \) does contain complete information about the function it was based on, and the Padé approach allows this information to be recovered.
Figure 3.7: Displays of the imaginary parts of truncated Taylor and corresponding Padé approximations of the Stieltjes function, starting from Taylor expansions using terms up thorough degrees 6 and 30, respectively.
### 3.3 Exercises

#### Exercise 3.3.1 Consider the Taylor expansion $f(z) = \sum_{n=0}^{\infty} (-z)^n$.

(a) Determine the radius of convergence $R$ for the Taylor series for $f(z)$.
(b) Sum the Taylor series in closed form.
(c) Based on the result in Part (b), show that the Taylor expansion of $f(z)$ around $z = \frac{1}{2}$ becomes $f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3}{2}\right)^n \left(z - \frac{1}{2}\right)^n$.
(d) From the expansion coefficients obtained in Part (c), determine the radius of convergence for the new expansion. Draw in the complex plane where the two expansions converge, and mark the point $z = -1$.

For parts (e) and (f) below, use only the original Taylor series coefficients for $f(z)$ (i.e. not the closed form expression for $f(z)$) to re-expand $f(z)$ around the point $z = \frac{1}{2}$. You can assume it known that $\sum_{k=m}^{\infty} (-1)^k \frac{k!}{m!(k-m)!} \frac{1}{2^m} = (-1)^m \left(\frac{3}{2}\right)^{m+1}$.

(e) Obtain the new Taylor coefficients by working out the values for $\frac{d^n}{dz^n} f(z)$, $n = 0, 1, 2, \ldots$ at $z = \frac{1}{2}$.

(f) Substitute $z \to \zeta + \frac{1}{2}$ in the original Taylor series, re-sort the result into powers of $\zeta$ and then change variable back $\zeta \to z - \frac{1}{2}$. 

#### Example 3.2.13 Substitute $f(k) = 1$, $k = 0, 1, 2, \ldots$ into Ramanujan’s formula (3.15).

Obviously, $f(z) \equiv 1$ will satisfy the given data (and also the growth condition for the right half plane). Equation (3.15) then gives the relation

$$\int_0^{\infty} t^{z-1} \frac{dt}{1+t} = \frac{\pi}{\sin \pi z}.$$ 

#### Example 3.2.14 Substitute $f(k) = \frac{1}{k!}$, $k = 0, 1, 2, \ldots$ into (3.15).

We arrive similarly at

$$\int_0^{\infty} e^{-t} t^{z-1} dt = \frac{\pi}{\Gamma(1-z) \sin \pi z}.$$ 

Since we recognize the integral as $\Gamma(z)$, we have arrived at (3.3). 

An alternative version of Ramanujan’s formula (with more rapid convergence of the series inside the integral) is

$$\Gamma(z) f(-z) = \int_0^{\infty} t^{z-1} \left( f(0) - \frac{t}{1!} f(1) + \frac{t^2}{2!} f(2) - + \ldots \right) dt$$

#### 3.2.9 Ramanujan’s formula

Analytic continuation of a Taylor expansion around $z = 0$ can be viewed as reconstructing a function globally based on knowing $f(0), f'(0), f''(0), f'''(0), \ldots$. If we instead know the function values $f(z)$ at an infinite sequence of points $z_k$, $k = 1, 2, 3, \ldots$ which has a finite limit point, there is in theory enough information available to extract all derivatives at this point. If the infinite sequence of points does not have a finite limit point, there will in general not be a unique solution. However, the following formula by Ramanujan may nevertheless be applicable - here given for the case that the values for $f(0), f(1), f(2), f(3), \ldots$ are known:

$$\frac{\pi}{\sin \pi z} f(-z) = \int_0^{\infty} t^{z-1} \left( f(0) - t f(1) + t^2 f(2) - + \ldots \right) dt$$

(3.15)

(where we may need analytic continuation to interpret the sum $f(0) - t f(1) + t^2 f(2) - + \ldots$ over the full interval $0 \leq t < \infty$).

It is easy to see that both $f(z) \equiv 0$ and $f(z) = \sin \pi z$ satisfy $f(0) = f(1) = f(2) = \ldots = 0$. However, if it is also known that $|f(z)| < C e^{A|z|}$ with $A < \pi$ holds when $\text{Re} z > 0$, it can be shown that (3.15) uniquely determines $f(z)$. Later in this text, Exercise 10.6.3 outlines a (somewhat heuristic) argument that leads to (3.15).

In the two examples below, we use the formula somewhat differently - enter known functions and obtain nontrivial integral formulas.

**Example 3.2.13** Substitute $f(k) = 1$, $k = 0, 1, 2, \ldots$ into Ramanujan’s formula (3.15).

Obvious, $f(z) \equiv 1$ will satisfy the given data (and also the growth condition for the right half plane). Equation (3.15) then gives the relation

$$\int_0^{\infty} \frac{t^{z-1}}{1+t} dt = \frac{\pi}{\sin \pi z}.$$ 

**Example 3.2.14** Substitute $f(k) = \frac{1}{k!}$, $k = 0, 1, 2, \ldots$ into (3.15).

We arrive similarly at

$$\int_0^{\infty} \frac{e^{-t} t^{z-1} dt}{\Gamma(1-z) \sin \pi z}.$$ 

Since we recognize the integral as $\Gamma(z)$, we have arrived at (3.3).
Exercise 3.3.2 Consider the function \( f(z) = \sum_{k=0}^{\infty} z^{(2^k)} \),
(a) Determine the radius of convergence of the Taylor expansion for \( f(z) \).
(b) Show that \( f(z) \) satisfies the functional equation
\[
f(z) = z + f(z^2).
\]
(c) By means of the functional equation, show that the function cannot be continued outside the Taylor series radius of convergence (i.e. its boundary forms a natural boundary for the function itself). Hint: From the functional equation, first deduce that \( f(z) = f(z^{2m}) + \sum_{k=0}^{m-1} z^{(2^k)} \) holds for \( m = 1, 2, 3, \ldots \) The result then follows by a similar argument to the one used in Example 3.1.2.

Exercise 3.3.3 This exercise supplements Example 3.2.12:
(a) Determine the radius of convergence of the Taylor series
\[
f(z) = \sum_{k=0}^{\infty} (-1)^k k! z^k. \tag{3.17}
\]
(b) Use Borel summation to obtain from (3.17) the formula
\[
f(z) = \int_0^{\infty} \frac{e^{-t}}{1 + 2t} \, dt, \tag{3.18}
\]
and determine the range of validity for this integral representation of \( f(z) \).
(c) Use instead the method from Section 3.2.5 to obtain (3.18) from (3.17). Hint: Note from the definition of the Gamma function that \( k! = \int_0^{\infty} e^{-t} t^k \, dt \).
(d) Starting from (3.18), derive (3.17) by repeated integration by parts.
(e) Again, start from (3.18) and arrive at (3.17) by the different method of first showing that (3.18) implies that \( f(z) \) satisfies the ODE
\[
z^2 f'(z) + (1 + z)f(z) - 1 = 0 \tag{3.19}
\]
with initial condition \( f(0) = 1 \). Then assume that \( f(z) \) has a Taylor expansion
\[
f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots
\]
and use (3.19) to determine the unknown coefficients.

Exercise 3.3.4
(a) By a change of variable in the integral definition of the Gamma function, derive
\[
\frac{1}{n^z} = \frac{1}{\Gamma(z)} \int_0^{\infty} t^{z-1} e^{-nt} \, dt.
\]
(b) Following one of the analytic continuation methods, show that the Riemann zeta-function
\[
\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}
\]
can be rewritten as
\[
\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1} \, dt.
\]
(c) Determine for what values of \( z \) the integral in part (b) converges.
Exercise 3.3.5  In one of the most famous letters in the history of mathematics (dated January 16, 1913), the Indian mathematical genius Srinivasa Ramanujan (1887-1920) introduces himself to the English mathematician G. H. Hardy (1877-1947), and then gives some stunning examples of original formulas. The letter also contains the seemingly whimsical statements

\[ 1 - 1! + 2! - 3! + 4! - + \ldots \approx 0.596 \]

and

\[ 1 + 2 + 3 + 4 + \ldots = -\frac{1}{12}. \]

Based on the examples in this Chapter 3, explain what lies behind both of these results.