Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
January 2024

Instructions:

Do two of three problems in each section (Prob and Stat).
Place an **X** on the lines next to the problem numbers
that you are **NOT** submitting for grading.

Prob
1. ____
2. ____
3. ____

Do not write your name anywhere on this exam.
You will be identified only by your student number.
Write this number **on each page** submitted for grading.
Show all relevant work!

Stat
4. ____
5. ____
6. ____
Total ____

Student Number ______________________

Probability Section

Problem 1.

We consider a sequence $\{X_i\}$, $i \geq 1$, of independent random variables such that for each positive integer $i \geq 2$,

$$\text{Prob}\left(X_i = -i\right) = \text{Prob}\left(X_i = i\right) = \frac{1}{2i \log i}, \qquad \text{and} \qquad \text{Prob}\left(X_i = 0\right) = 1 - \frac{1}{i \log i},$$

with

$$\text{Prob}\left(X_1 = 0\right) = 1.$$

Let

$$S_n = \sum_{i=1}^{n} X_i,$$

(a) (20 points.) Prove that X_i obeys the weak law of large numbers, to wit

$$\forall \varepsilon > 0, \lim_{n\to\infty} \text{Prob}\left(\left|\frac{S_n}{n}\right| \geq \varepsilon\right) = 0.$$

We will now prove that X_i does not obey the strong law of large numbers. By contradiction, assume that $n^{-1}\sum_{i=1}^{n} X_i \to 0$ almost surely.

(b) (30 points.) Let Ω_i be the event $\left|\frac{X_i}{i}\right| \geq 1$. Deduce that

$$\text{Prob}\left(\bigcap_{i_0=1}^{\infty} \bigcup_{i=i_0}^{\infty} \Omega_i\right) = 0.$$

(c) (20 points.) Prove that

$$\lim_{n\to\infty} \sum_{i=1}^{n} P(\Omega_i) = \infty.$$

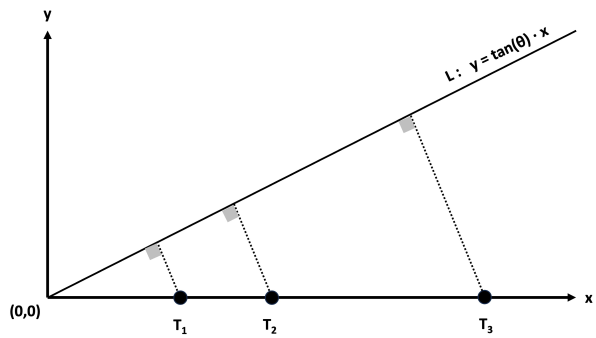
(d) (30 points.) Prove that

$$\text{Prob}\left(\bigcap_{i_0=1}^{\infty} \bigcup_{i=i_0}^{\infty} \Omega_i \right) = 1,$$

and conclude. **Hint.** You may find useful to use that $(1-x) \leq e^{-x}$, for $x \geq 0$.

Problem 2.

In the xy-plane, consider the arrival times $\{T_n\}_{n\geq 0}$ of a homogeneous Poisson point process (HPPP) with intensity $\lambda > 0$ along the non-negative x-axis. Imagine projecting the process' arrivals orthogonally along the line $L: \ y = \tan(\theta) \cdot x$, for a given $0 < \theta < \pi/2$ (see the figure below).



Let D_n denote the distance between $(0,0)$ and the orthogonal projection of T_n along L.

(a) (30 points.) Determine whether $D := \{D_n\}_{n\geq 1}$ is an HPPP and, if so, its intensity. Justify your answer!

(b) (10 points.) What's the distribution and expected value of $(D_4 - D_1)$?

(c) (10 points.) What's the probability of the event $A := [0 < D_1 \leq D_2 \leq D_3 < 1 < D_4]$?

(d) (10 points.) Given the event A, what's the conditional covariance between D_4 and D_1?

(e) (40 points.) Given the event A, what's the conditional expectation of $(D_4 - D_1)$?

Problem 3.

Consider a time-homogeneous Markov process $X = (X_t)_{t\geq 0}$ with state space

$$S = \{0, 1, 2, \ldots\},$$

and rate matrix Q which satisfies $Q(i,j) = 0$ when $|i - j| > 1$.

(a) (50 points.) Let $\pi : S \to \mathbb{R}$ be a function with $\pi(i) \geq 0$, for all $i \in S$, and $\sum_{i\in S} \pi(i) = 1$. Show that π is a stationary distribution of X if and only if π satisfies the detailed-balance condition.

(b) (50 points.) Next, apply the above to address the stationarity of an M/M/3 queue. In particular, X keeps track of the number of customers in line or being served in a queue with an external arrival rate $\lambda > 0$ and three servers, each of which has a service rate $\mu > 0$. Determine a necessary and sufficient condition for this queue to have a stationary distribution, and provide this distribution if the condition is met.

Statistics Section

Problem 4.

Let $X_1, \ldots, X_n$, with $n \geq 3$, be a random sample from a Poisson distribution with an unknown rate parameter $\lambda > 0$.

(a) (30 points.) Determine the expected value of $X_1 \cdot I[X_2 = 0]$, where $I[A]$ is the indicator of an event A.

(b) (70 points.) Determine explicitly the uniformly minimum variance unbiased estimator (UMVUE) of $\lambda e^{-\lambda}$.

Problem 5.

Let $X_1, \ldots, X_n$ be a random sample from the Uniform$[0, \theta]$ distribution, where $\theta > 0$ is unknown.

(a) (30 points.) Determine a 99% confidence lower-bound for θ based on $X_{(n)} := \max\{X_1, \ldots, X_n\}$.

(b) (30 points.) Show that $X_{(n)}$ is consistent for θ.

(c) (40 points.) Determine the $\lim_{n\to\infty} n\big(\theta - X_{(n)}\big)$ <u>in distribution</u>.

Problem 6. (100 points.) Let X and Y be two independent random variables uniformly distributed on $[\theta, \theta + 1]$. To decide in favor of the hypothesis $H_0 : \theta = 0$ against the alternative $H_1 : \theta > 0$, we consider two tests defined by

$$\begin{aligned} \Phi_1(X) &: \ \text{reject } H_0 \text{ if } X > 0.95, \\ \Phi_2(X, Y) &: \ \text{reject } H_0 \text{ if } X + Y > C. \end{aligned}$$

(a) (5 points.) Find the value of C so that $\Phi_1(X)$ and $\Phi_2(X, Y)$ have the same size.

(b) (10 points.) Compute the power function of each test, and sketch both functions on the same figure.

(c) (5 points.) Prove or disprove that $\Phi_2(X, Y)$ is more powerful than $\Phi_1(X)$.

(d) (40 points.) Design a test $\Phi_3(X, Y)$ that has the same size as $\Phi_2(X, Y)$, but is more powerful than $\Phi_2(X, Y)$.

(e) (40 points.) Compute the power of $\Phi_3(X, Y)$.
