A Time-Inconsistent Dynkin Game:

Intra-personal v.s. Inter-personal Equilibria

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OPTIMAL CONTROL/STOPPING

- Consider a Markovian state process X.

Stochastic Optimal Control/Stopping

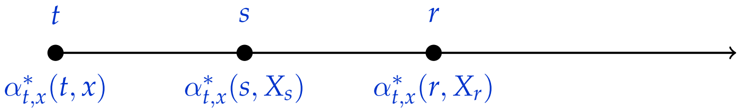
Given $(t, x) \in [0, \infty) \times \mathbb{R}^d$, can we solve

$$\sup_{\alpha \in \mathcal{A}} F(t, x, \alpha)? \tag{1}$$

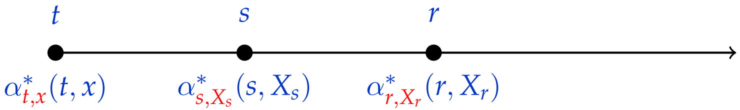
- **Classical Theory:**
  - **Want:** find an optimal strategy $\alpha^*_{t,x} \in \mathcal{A}$.
  - **Methods:** *dynamic programming, martingale approach,...*
  - Consider $\alpha^*_{t,x}$ as a mapping:

$$(t, x) \quad \longmapsto \quad \alpha^*_{t,x} \in \mathcal{A}.$$

- **Problem Solved.** *Feeling Good?*



- **The Reality:**



- **Time Inconsistency:**
  - $\alpha^*_{t,x}$, α^*_{s,X_s}, α^*_{r,X_r} may all be different.
  - The original objective (1) cannot be attained...

Time-inconsistent objectives:

- Non-exponential discounting:

$$F(t,x,\alpha) := \mathbb{E}_{t,x}[\delta(T-t)g(X_T^\alpha)].$$

- Payoff depending on initials (t,x):

$$F(t,x,\alpha) := \mathbb{E}_{t,x}[g(t,x,X_T^\alpha)].$$

- Nonlinear functionals of $\mathbb{E}[\cdot]$:

$$F(t,x,\alpha) := \mathbb{E}_{t,x}[g(X_T^\alpha)] - H(\mathbb{E}_{t,x}[g(X_T^\alpha)]).$$

- Probability distortion:

$$F(t,x,\alpha) := \int_0^\infty w\Big(\mathbb{P}_{t,x}\left[g(X_T^\alpha) > u\right]\Big)du.$$

How to resolve time inconsistency?

Consistent Planning [Strotz (1955-56)]

- Take into account future selves' behavior.

  Find an *(intra-personal) equilibrium* strategy that

  once being enforced over time,
  no future self would want to deviate from.

- **How to precisely define and characterize equilibria?**

  For control problems,

  - Ekeland & Lazrak (2006): Definition via spike variations.
  - Ekeland & Pirvu (2008): Extended HJB system characterizes equilibrium controls α^*.

Literature

- **Control problems:**

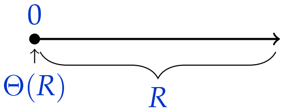
  Ekeland, Mbodji, & Pirvu (2012), Björk, Murgoci, & Zhou (2014), Dong & Sircar (2014), Björk & Murgoci (2014), Yong (2012), Björk, Khapko & Murgoci (2017), ...

- **Stopping problems:**
  (a) **Extensions from "control" to "stopping"**
    - Same *definition* and *extended HJB system* as in control case.
    - Christensen & Lindensjö (2018, 2020), Ebert et al. (2020).

  (b) **The fixed-point approach**
    - Equilibria as fixed-points, found via fixed-point iterations.
    - Huang & Nguyen-Huu (2018): non-exponential discounting; Huang, Nguyen-Huu, & Zhou (2020): probability distortion; Huang & Yu (2021): model uncertainty.

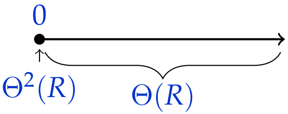
FIXED-POINT APPROACH

1. At first, one follows $R \in \mathcal{B}(\mathbb{R}^d)$.



$\Longrightarrow$ switch from R to $\Theta(R)$

2. Now, one follows $\Theta(R)$.



$\Longrightarrow$ switch from $\Theta(R)$ to $\Theta^2(R)$

3. Continue this procedure *until* we reach

$$R_0 := \lim_{n\to\infty} \Theta^n(R)$$

Expect: $\Theta(R_0) = R_0$, i.e. cannot improve anymore.

How about a game with "multiple agents"?

- Each agent has time-inconsistent preferences.
- Two levels of game-theoretic reasoning:

  Each agent...

  1. looks for an *intra-personal equilibrium* among her current and future selves;
  2. finds the best response to other agents' strategies.

- An *inter-personal equilibrium* in the game is then

  A **Nash equilibrium** among all agents,
  each of whom uses her *best* intra-personal equilibrium.

  - Such *inter-personal equilibrium* have not been studied. (Precise definition? Existence? Construction?)

IN THIS TALK...

Focus on a Dynkin game:

- $\mathbb{Z}_+ := \{0, 1, 2, ...\}$.
- $X = (X_t)_{t\in\mathbb{Z}_+}$ time-homogeneous strong Markov, taking values in $\mathbb{X}$.
- For $i \in \{1, 2\}$, given the other player uses $\sigma \in \mathcal{T}$, Player i maximizes

$$J_i(x, \tau, \sigma) := \mathbb{E}_x[F_i(\tau, \sigma)], \tag{2}$$

over $\tau \in \mathcal{T}$, where

$$F_i(\tau, \sigma) := \delta_i(\tau) f_i(X_\tau) 1_{\{\tau<\sigma\}} + \delta_i(\sigma) g_i(X_\sigma) 1_{\{\tau>\sigma\}} + \delta_i(\tau) h_i(X_\tau) 1_{\{\tau=\sigma\}}, \quad \forall \tau, \sigma \in \mathcal{T}.$$

  - $f_i, g_i, h_i : \mathbb{X} \to \mathbb{R}_+$ Borel measurable.
  - $\delta_i : \mathbb{R}_+ \to [0, 1]$ decreasing, $\delta_i(0) = 1$ (e.g. $\delta_i(t) = e^{-rt}$).

NON-EXPONENTIAL DISCOUNTING

Assume the discount function $\delta_i : \mathbb{R}_+ \to [0,1]$ satisfies

$$\delta_i(t)\delta_i(s) \leq \delta_i(t+s) \quad \forall\, t, s \in \mathbb{Z}_+. \tag{3}$$

- Captures *decreasing impatience* in behavioral economics.
- Examples:
  - hyperbolic $\delta_i(t) = \frac{1}{1+\beta t}$,
  - generalized hyperbolic $\delta_i(t) = \frac{1}{(1+\beta t)^k}$,
  - pseudo-exponential $\delta_i(t) = \lambda e^{-r_1 t} + (1-\lambda)e^{-r_2 t}$.
- Time inconsistency arises under (3).

STOPPING POLICIES

Assume each player stops at the first entrance time

$$\rho_S := \inf\{t \geq 0 : \; X_t \in S\}.$$

- $S \in \mathcal{B}$, a Borel subset of $\mathbb{X}$, is called a *stopping policy*.
- Given the other player using $T \in \mathcal{B}$, Player i's intra-personal reasoning

$$\begin{aligned}\Theta_i^T(S) :=& \{x \in S : J_i(x, 0, \rho_T) \geq J_i(x, \rho_S^+, \rho_T)\} \\ & \cup \{x \notin S : J_i(x, 0, \rho_T) > J_i(x, \rho_S^+, \rho_T)\} \in \mathcal{B}. \quad (4)\end{aligned}$$

  - Stop at time 0 $\implies J_i(x, 0, \rho_T)$.
  - Don't stop at time 0 $\implies J_i(x, \rho_S^+, \rho_T)$, where

$$\rho_S^+ := \inf\{t > 0 : \; X_t \in S\}.$$

Intra- and Inter-personal Equilibria

Definition (Intra-personal)

$S \in \mathcal{B}$ is Player i's *intra-personal equilibrium* w.r.t. $T \in \mathcal{B}$ if

$$\Theta_i^T(S) = S.$$

We denote this by $S \in \mathcal{E}_i^T$.

Definition (*Soft* inter-personal)

$(S, T) \in \mathcal{B} \times \mathcal{B}$ is a <u>*soft*</u> *inter-personal equilibrium* if

$$\Theta_1^T(S) = S \quad \text{and} \quad \Theta_2^S(T) = T.$$

We denote this by $(S, T) \in \mathcal{E}$.

Given $T \in \mathcal{B}$, define the *value function* of $S \in \mathcal{E}_i^T$ by

$$U_i^T(x, S) := J_i(x, 0, \rho_T) \vee J_i(x, \rho_S^+, \rho_T), \quad x \in \mathbb{X}.$$

Definition (*Optimal* Intra-personal)

$S \in \mathcal{E}_i^T$ is Player i's *optimal intra-personal equilibrium* w.r.t. $T \in \mathcal{B}$ if,

$$\text{for any } R \in \mathcal{E}_i^T, \quad U_i^T(x, S) \geq U_i^T(x, R) \ \forall\, x \in \mathbb{X}.$$

We denote this by $S \in \widehat{\mathcal{E}}_i^T$.

- "Optimal equilibrium" of Huang & Zhou (2019):
  - Wants an equilibrium to be uniformly dominating —a rare occurrence in game theory.
  - For stopping under (3), optimal equilibrium exists. (Huang & Zhou (2019, 2020), Huang & Wang (2021))

Definition (*Sharp* inter-personal)

$(S, T) \in \mathcal{B} \times \mathcal{B}$ is a *sharp inter-personal equilibrium* if

$$S \in \widehat{\mathcal{E}}_1^T \quad \text{and} \quad T \in \widehat{\mathcal{E}}_2^S.$$

We denote this by $(S, T) \in \widehat{\mathcal{E}}$.

Ultimate goals:

- Existence of *sharp* inter-personal equilibria.
- Construction via concrete *iterative procedures*.

First Question: *What type of iterations to use?*

- Fixed-point iteration, i.e.

$$\lim_{n\to\infty}(\Theta_i^T)^n(S) \in \mathcal{E}_i^T$$

  does not seem so promising...

- **Any iteration that directly leads to $S^* \in \widehat{\mathcal{E}}_i^T$?**
  - Recently approached by Bayraktar, Zhang, & Zhou (2020) in a one-player stopping poblem.

For any $T \in \mathcal{B}$, define $\Phi_i^T : \mathcal{B} \to \mathcal{B}$ by

$$\Phi_i^T(S) := S \cup \left\{ x \notin S : J_i(x, 0, \rho_T) > V_i^T(x, S) \right\}, \tag{5}$$

where

$$V_i^T(x, S) := \sup_{1 \le \tau \le \rho_S^+} \mathbb{E}_x[F_i(\tau, \rho_T)] \quad x \in \mathbb{X}, S \in \mathcal{B}. \tag{6}$$

Theorem (Direct iteration to $\widehat{\mathcal{E}}_i^T$)

Assume $\boxed{h_i \le g_i}$. *Given* $T \in \mathcal{B}$, *define* $\left(S_i^n(T)\right)_{n \in \mathbb{N}} \subset \mathcal{B}$ *by*

$$S_i^1(T) := \Phi_i^T(\emptyset), \quad S_i^n(T) := \Phi_i^T\left(S^{n-1}(T)\right) \text{ for } n \ge 2, \tag{7}$$

Then,

$$\Gamma_i(T) := \bigcup_{n \in \mathbb{N}} S_i^n(T) \in \widehat{\mathcal{E}}_i^T. \tag{8}$$

Alternating Iteration

Let Players 1 and 2 *take turns* to perform iteration (7).

$$
\begin{array}{ll}
S_0 & T_0 = \Gamma_2(S_0) \\
S_1 = \Gamma_1(T_0) & T_1 = \Gamma_2(S_1) \\
S_2 = \Gamma_1(T_1) & T_2 = \Gamma_2(S_2) \\
\vdots & \vdots
\end{array}
$$

Hope:

1) (S_n, T_n) converges appropriately.
2) The limit (S_∞, T_∞) is a (*sharp*) inter-personal equilibrium.

Lemma

Assume $\boxed{f_i \leq h_i \leq g_i}$ *and*

$$\boxed{(\delta_i(t) g_i(X_t^x))_{t \geq 0} \text{ is a supermartingale} \quad \forall x \in \mathbb{X}.} \tag{9}$$

Then, for any $T, R \in \mathcal{B}$ *with* $T \subseteq R$,

$$\Phi_i^T(S) \supseteq \Phi_i^R(S) \supseteq \Phi_i^R(S') \quad \forall S, S' \in \mathcal{B} \text{ with } S \supseteq S'. \tag{10}$$

Proof Sketch. (9) implies

$$J_i(x, \tau, \rho_T) \leq J_i(x, \tau, \rho_R) \quad \forall x \in \mathbb{X},\ \tau \in \mathcal{T}. \tag{11}$$

In view of (6), for any $x \in \mathbb{X}$ and $S, S' \in \mathcal{B}$ with $S \supseteq S'$,

$$V_i^T(x, S) \leq V_i^R(x, S) \leq V_i^R(x, S'). \implies \Phi_i^T(S) \supseteq \Phi_i^R(S) \supseteq \Phi_i^R(S').$$

Corollary ($T \mapsto \Gamma_i(T)$ monotone)

Assume $\boxed{f_i \le h_i \le g_i}$ *and* $\boxed{(9)}$. *For any* $T, R \in \mathcal{B}$ *with* $T \subseteq R$,

$$\Gamma_i(T) \supseteq \Gamma_i(R).$$

Idea: Taking $S_0 = \emptyset$,

$$\begin{aligned} S_0 &= \emptyset & T_0 &= \Gamma_2(S_0) \\ S_1 &= \Gamma_1(T_0) & T_1 &= \Gamma_2(S_1) \\ S_2 &= \Gamma_1(T_1) & T_2 &= \Gamma_2(S_2) \\ &\vdots & &\vdots \end{aligned}$$

Hence, the limit is well-defined as

$$(S_\infty, T_\infty) = (\cup_n S_n, \cap_n T_n) \in \mathcal{B} \times \mathcal{B}.$$

Theorem (Existence of the *soft*)

Assume $\boxed{f_i \le h_i \le g_i}$ *and* $\boxed{(9)}$. *Set* $S_0 := \emptyset$ *and define*

$$T_n := \Gamma_2(S_n) \qquad S_{n+1} := \Gamma_1(T_n), \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{12}$$

Then, $(S_\infty, T_\infty) := (\cup_n S_n, \cap_n T_n) \in \mathcal{E}$ *and satisfies*

$$\Gamma_1(T_\infty) = S_\infty, \quad \Gamma_2(S_\infty) \subseteq T_\infty. \tag{13}$$

Proof. Fix $x \in S_\infty$. $\exists N \in \mathbb{N}$ s.t. $x \in S_{n+1} = \Gamma_1(T_n) \in \mathcal{E}_1^{T_n}$ $\forall n > N$.

$$J_1(x, 0, \rho_{T_n}) \ge J_1(x, \rho^+_{S_{n+1}}, \rho_{T_n}) \;\; \forall n \ge N.$$

$$\Longrightarrow J_1(x, 0, \rho_{T_\infty}) \ge J_1(x, \rho^+_{S_\infty}, \rho_{T_\infty}), \text{ i.e. } x \in \Theta_1^{T_\infty}(S_\infty).$$

Thus, $S_\infty \subseteq \Theta_1^{T_\infty}(S_\infty)$. Can get $(S_\infty)^c \subseteq (\Theta_1^{T_\infty}(S_\infty))^c$ similarly.

Conclude: $S_\infty = \Theta_1^{T_\infty}(S_\infty)$ and $T_\infty = \Theta_2^{S_\infty}(T_\infty)$.

Proof (conti.). By monotonicity of $T \mapsto \Gamma_i(T)$,

$$S_n \subseteq S_\infty \implies \Gamma_2(S_\infty) \subseteq \Gamma_2(S_n) = T_n \implies \Gamma_2(S_\infty) \subseteq T_\infty,$$
$$T_n \supseteq T_\infty \implies \Gamma_1(T_\infty) \supseteq \Gamma_1(T_n) = S_{n+1} \implies \Gamma_1(T_\infty) \supseteq S_\infty,$$

Also, by $S_0 = \emptyset \subseteq S_\infty \in \mathcal{E}_1^{T_\infty}$, can construct $\{S_i^n(T_\infty)\}_n$ in (7) and find $S_i^n(T_\infty) \subseteq S_\infty$ for all n. Hence, $\Gamma_1(T_\infty) \subseteq S_\infty$.

Lemma

Assume $\boxed{h_i \leq g_i}$. *If* $(S, T) \in \mathcal{B} \times \mathcal{B}$ *satisfies*

$$\Gamma_1(T) = S \quad \text{and} \quad \Gamma_2(S) = T, \tag{14}$$

then $(S, T) \in \widehat{\mathcal{E}}$.

- By (13), (S_∞, T_∞) is *almost* sharp!

EXAMPLE I

- $\mathbb{X} = \{x_0, x_1, x_2, \dots\}$ with

$$\mathbb{P}_{x_{n+1}}(X_1 = x_n) = 1, \quad n = 0, 1, 2 \dots,$$
$$\mathbb{P}_{x_0}(X_1 = x_0) = 1 - \varepsilon, \ \mathbb{P}_{x_0}(X_1 = x_1) = \varepsilon, \ \text{for } \varepsilon \in [0, 1).$$

- Take $M > 1$ such that $\delta_2(2) < 1/M < \delta_2(1)$.
- Take $L > 1$ and define

$$f_1(x_n) = 1, \ g_1(x_n) = L \quad n = 0, 1, 2, \dots,$$
$$f_2(x_0) = 0, \ f_2(x_n) = 1 \ \ n = 1, 2, \dots, \ g_2(x_n) = M \ \ n = 0, 1, 2 \dots,$$

while h_i is any function such that $f_i \le h_i \le g_i$.

For $\varepsilon \in [0,1)$ small enough,

$$\begin{aligned} S_0 &= \emptyset, & T_0 &= \{x_1, x_2, \dots\}, \\ S_1 &= \{x_0\}, & T_1 &= \{x_2, x_3, \dots\}, \\ S_2 &= \{x_0, x_1\}, & T_2 &= \{x_3, x_4, \dots\}, \\ &\vdots & &\vdots \\ S_n &= \{x_0, x_1, \dots, x_{n-1}\}, & T_n &= \{x_{n+1}, x_{n+2}, \dots\}. \end{aligned}$$

- $(S_\infty, T_\infty) = (\mathbb{X}, \emptyset)$.
- Is it *sharp*? Let's check $\Gamma_2(\mathbb{X}) = \emptyset$.

$$V_2^{\mathbb{X}}(x_n, \emptyset) = \sup_{1 \le \tau \le \rho_\emptyset^+} \mathbb{E}_{x_n}[F_2(\tau, \rho_{\mathbb{X}})] = g_2(x_n) \ge h_2(x_n) = J_2(x_n, 0, \rho_{\mathbb{X}}).$$

This implies $\Phi_2^{\mathbb{X}}(\emptyset) = \emptyset$, so $\Gamma_2(\mathbb{X}) = \emptyset$. $\implies (S_\infty, T_\infty) \in \widehat{\mathcal{E}}$.

EXAMPLE II

- $\mathbb{X} = \{x_0, x_1, x_2, \dots\} \cup \{y, z\}$.
- All previous settings remain.
- Transition probabilities for $\{y, z\}$

$$\mathbb{P}_y(X_1 = x_n) = p_n > 0 \text{ with } \sum_{n=0}^{\infty} p_n = 1, \;\; \mathbb{P}_z(X_1 = y) = 1.$$

- $\delta_2(1)^2 < \delta_2(2)$.
- Payoffs on $\{y, z\}$:

$$f_2(y) = M\delta_2(1), \quad f_2(z) \in \left(M\delta_2(1)^2 \vee \delta_2(2), M\delta_2(2)\right), \\ g_2(y) = g_2(z) = M.$$

  - Only require $f_1 \leq g_1$.

For $\varepsilon \in [0,1)$ small enough,

$$S_0 = \emptyset, \qquad T_0 = \{x_1, x_2, \dots\} \cup \{y, z\},$$
$$S_1 = \{x_0\}, \qquad T_1 = \{x_2, x_3, \dots\} \cup \{y, z\},$$
$$S_2 = \{x_0, x_1\}, \qquad T_2 = \{x_3, x_4, \dots\} \cup \{y, z\},$$
$$\vdots \qquad \vdots$$
$$S_n = \{x_0, x_1, \dots, x_{n-1}\}, \qquad T_n = \{x_{n+1}, x_{n+2}, \dots\} \cup \{y, z\}.$$

- $(S_\infty, T_\infty) = (\mathbb{X}, \{y, z\})$.
- Similarly to Example I, $\Gamma_2(\mathbb{X}) = \emptyset$. $\implies$ $\Gamma_2(S_\infty) = \emptyset \subsetneq T_\infty$.
- **Note:** $\emptyset = \Gamma_2(S_\infty) \in \widehat{\mathcal{E}}_2^{S_\infty}$ dominates $T_\infty \in \mathcal{E}_2^{S_\infty}$ at z:

$$U_2^{S_\infty}(z, T_\infty) = J_2(z, 0, \rho_{S_\infty}) \vee J_2(z, \rho^+_{T_\infty}, \rho_{S_\infty}) = f_2(z) \vee M\delta_2(1)^2$$
$$= f_2(z) < M\delta_2(2) = J_2(z, \rho^+_\emptyset, \rho_{S_\infty}) \leq U_2^{S_\infty}(z, \emptyset).$$

- So, $T_\infty \notin \widehat{\mathcal{E}}_2^{S_\infty}$. $\implies$ $(S_\infty, T_\infty) \notin \widehat{\mathcal{E}}$.

Existence of the *Sharp*

Assumption 1

X has transition densities $(p_t)_{t\geq 1}$ w.r.t a measure μ on $(\mathbb{X}, \mathcal{B})$. That is, for $t = 1, 2, ..., p_t : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ is Borel and

$$\mathbb{P}_x(X_t \in A) = \int_A p_t(x, y)\mu(dy) \quad \forall x \in \mathbb{X},\ A \in \mathcal{B}.$$

Lemma

Let μ be a measure on $(\mathbb{X}, \mathcal{B})$. Given $A \subseteq \mathbb{X}$, there is a maximal Borel minorant of A under μ, defined as

- a set $A^\mu \in \mathcal{B}$ with $A^\mu \subseteq A$ such that

$$\text{for any } A' \in \mathcal{B} \text{ with } A' \subseteq A, \quad \mu(A' \setminus A^\mu) = 0.$$

Theorem (Existence of the *sharp*)

Under $\boxed{\text{Assumption 1}}$, $\boxed{f_i \leq h_i \leq g_i}$, and $\boxed{(9)}$,

there exists a *sharp* inter-personal equilibrium.

To prove this, we focus on the collection

$$A := \{(S, T) \in \mathcal{E} : \Gamma_1(T) \supseteq S \text{ and } \Gamma_2(S) \subseteq T\}.$$

By previous Thm, $A \neq \emptyset$. Define a *partial order* on A:

$$(S, T) \succeq (S', T') \quad \text{if } S \supseteq S' \text{ and } T \subseteq T'. \tag{15}$$

PROOF (STEP 1)

Suppose there exists a *maximal element* $(\bar{S}, \bar{T})$ in A.

Claim: $(\bar{S}, \bar{T}) \in \widehat{\mathcal{E}}$.

- Set $S_0 := \bar{S}$, $T_0 := \bar{T}$. Do alternating iteration:

$$S_{n+1} := \Gamma_1(T_n) \quad \text{and} \quad T_{n+1} := \Gamma_2(S_{n+1}) \quad \forall n \geq 0.$$

- As shown in previous Thm,

$$(S_\infty, T_\infty) := (\cup_n S_n, \cap_n T_n) \in A.$$

- By construction, $S_\infty \supseteq S_0 = \bar{S}$ and $T_\infty \subseteq T_0 = \bar{T}$.
As $(\bar{S}, \bar{T})$ is maximal in A, $S_\infty = S_0 = \bar{S}$ and $T_\infty = T_0 = \bar{T}$.

$$\begin{cases} \Gamma_1(\bar{T}) = \Gamma_1(T_0) = S_1 = S_0 = \bar{S} \\ \Gamma_2(\bar{S}) = \Gamma_2(S_0) = \Gamma_2(S_1) = T_1 = T_0 = \bar{T} \end{cases} \implies (\bar{S}, \bar{T}) \in \widehat{\mathcal{E}}.$$

Proof (Step 2)

Let $(S_\alpha, T_\alpha)_{\alpha\in I}$ be a *totally ordered* subset of A.

Claim: $(S_\alpha, T_\alpha)_{\alpha\in I}$ has an upper bound in A.

Idea:

- Set $S_0 := \cup_{\alpha\in I} S_\alpha$, $T_0 := \cap_{\alpha\in I} T_\alpha$. Do alternating iteration:

$$S_{n+1} := \Gamma_1(T_n) \quad \text{and} \quad T_{n+1} := \Gamma_2(S_{n+1}) \quad \forall n \geq 0.$$

- Expect: $(S_\infty, T_\infty) := (\cup_n S_n, \cap_n T_n) \in A$.
  $\implies$ This is an upper bound for $(S_\alpha, T_\alpha)_{\alpha\in I}$.

Measurability issue: $\cup_{\alpha\in I} S_\alpha, \cap_{\alpha\in I} T_\alpha \notin \mathcal{B}$ in general!

Proof (Step 2)

- Let $T_0^\mu \in \mathcal{B}$ a maximal Borel minorant of T_0 under μ.
  - $T_0^\mu \in \mathcal{B}$ and $T_0^\mu \subseteq T_0$.
  - For any $T \in \mathcal{B}$ with $T \subseteq T_0$, $\mu(T \setminus T_0^\mu) = 0$.
- For any $T \in \mathcal{B}$ with $T \subseteq T_0$,

$$\mathbb{P}_x(X_t \in T \setminus T_0^\mu) = \int_{T \setminus T_0^\mu} p_t(x, y)\mu(dy) = 0 \quad \forall x \in \mathbb{X},\ t > 0.$$

Hence,

$$\mathbb{P}_x(X_t \in T \setminus T_0^\mu \text{ for some } t \in \mathbb{N}) = 0 \quad \forall x \in \mathbb{X}. \tag{16}$$

Proof (Step 2)

- *Modified* alternating iteration:

$$S_0 = \cup_{\alpha \in I} S_\alpha \qquad T_0 = \cap_{\alpha \in I} T_\alpha$$
$$S_1 = \Gamma_1(T_0^\mu) \supseteq S_0 \qquad T_1 = \Gamma_2(S_1)$$
$$S_2 = \Gamma_1(T_1) \qquad T_2 = \Gamma_2(S_2)$$
$$\vdots \qquad \vdots$$

- For any $T \in \mathcal{B}$ with $T \subseteq T_0$,

$$\Gamma_1(T) \supseteq \Gamma_1(T_\alpha) \supseteq S_\alpha \;\; \forall \alpha \in I \implies \Gamma_1(T) \supseteq S_0.$$

For any $S \in \mathcal{B}$ with $S \supseteq S_0$,

$$\Gamma_2(S) \subseteq \Gamma_2(S_\alpha) \subseteq T_\alpha \;\; \forall \alpha \in I \implies \Gamma_2(S) \subseteq T_0.$$

PROOF (STEP 2)

- *Modified* alternating iteration:

$$\begin{aligned} S_0 &= \cup_{\alpha\in I} S_\alpha & T_0 &= \cap_{\alpha\in I} T_\alpha \\ S_1 &= \Gamma_1(T_0^\mu) \supseteq S_0 & T_1 &= \Gamma_2(S_1) \subseteq T_0 \\ S_2 &= \Gamma_1(T_1) \supseteq \Gamma_1(T_0) \neq S_1 & T_2 &= \Gamma_2(S_2) \\ &\vdots & &\vdots \end{aligned}$$

- By (16), $\rho_{T_1\cup T_0^\mu} = \rho_{T_0^\mu}$ $\mathbb{P}_x$-a.s., for $x \notin T_1 \cup T_0^\mu$.

$$S_2 := \Gamma_1(T_1) \supseteq \Gamma_1(T_1 \cup T_0^\mu) = \Gamma_1(T_0^\mu) = S_1.$$

PROOF (STEP 2)

- *Modified* alternating iteration:

$$S_0 = \cup_{\alpha \in I} S_\alpha \qquad T_0 = \cap_{\alpha \in I} T_\alpha$$
$$S_1 = \Gamma_1(T_0^{\mu}) \supseteq S_0 \qquad T_1 = \Gamma_2(S_1) \subseteq T_0$$
$$S_2 = \Gamma_1(T_1) \supseteq S_1 \qquad T_2 = \Gamma_2(S_2) \subseteq T_1$$
$$\vdots \qquad \vdots$$

Conclude: $(S_\infty, T_\infty) := (\cup_n S_n, \cap_n T_n) \in A$ is well-defined, and is an upper bound for $(S_\alpha, T_\alpha)_{\alpha \in I}$.

- By Zorn's lemma, the proof is complete.

Summary

- **Soft inter-personal equilibrium:**

  A **Nash equilibrium** between two players,
  each of whom uses an intra-personal equilibrium.

  - Always exists.
  - Can be found via concrete alternating iteration.

- **Sharp inter-personal equilibrium:**

  A **Nash equilibrium** between two players,
  each of whom uses an *optimal* intra-personal equilibrium.

  - Exists, if X has transition densities.
  - Constructed via alternating iteration + Zorn's lemma.

APPLICATION TO NEGOTIATION

- **Firms 1, 2** want to cooperate to initiate a project
  - Each firm has a proprietary skill/technology.
  - Revenue $R > 0$ fixed.
  - Cost $X > 0$ is random: $\exists\, u > 1$ and $p \in (0, 1)$ s.t.

$$\mathbb{P}_x[X_1/x = u] = p \text{ and } \mathbb{P}_x[X_1/x = 1/u] = 1 - p, \quad \forall x \in \mathbb{X}.$$

That is, X evolves on the binomial tree

$$\mathbb{X} = \{u^i : i = 0, \pm 1, \pm 2, \dots\} \tag{17}$$

  - Assume: X is a submartingale, i.e. $p \geq \frac{1}{u+1}$.
- **Each firm** insists on...
  - Taking a larger (risk-free) share $N \in (R/2, R)$;
  - Demanding the other to take smaller share $K := R - N \in (0, R/2)$ **and** incur (risky) cost X.

- In our Dynkin game,

$$F_i(\tau,\sigma) := \delta_i(\tau)(K - X_\tau)^+ 1_{\{\tau<\sigma\}} + \delta_i(\sigma) N 1_{\{\tau>\sigma\}} + \delta_i(\tau) h_i(X_\tau) 1_{\{\tau=\sigma\}},$$

  - $\tau < \sigma$: Firm i gives in first.
  - $\tau > \sigma$: The other firm gives in first.
  - $f_1(x) = f_2(x) = (K - x)^+$, $g_1(x) = g_2(x) = N$, $f_i \le h_i \le g_i$.
  - Hyperbolic discounting:

$$\delta_i(t) = \frac{1}{1 + \beta_i t},$$

    - $\beta_i > 0$: impatience level of Firm i.

THE STRATEGY OF COERCION

- Demonstrate (or pretend!?) a strong will *not* to give in...
  - ...to *coerce* the other firm to give in.
  - "Never give in" $\iff \tau = \infty \iff S_0 = \emptyset$.

Proposition (Firm 1 *more patient*)

If $\beta_1 \leq \beta_2$, the alternating iterative procedure (12) terminates after one iteration: $\exists\, y_2^* \in [0, \infty) \cap \mathbb{X}$ s.t.

$$\underset{\sim\sim\sim}{S_0 = \emptyset} \implies T_0 = (0, y_2^*] \cap \mathbb{X} \implies \underset{\sim\sim\sim}{S_1 = \emptyset}.$$

Moreover, $(S_\infty, T_\infty) = (S_0, T_0) = (\emptyset, (0, y_2^*] \cap \mathbb{X}) \in \widehat{\mathcal{E}}$.

- **Message:** *"More patient"* $\implies$ coercion works

- What if Firm 1 is *less patient* ($\beta_1 > \beta_2$)?
  - Complicated...
    $\Longrightarrow$ Coercion may or may not work.

Proposition (Firm 1 *significantly less patient*)

If $\beta_1 > 0$ sufficiently large and $\beta_2 > 0$ sufficiently small, iterative procedure (12) yields

$$(\underline{\underline{S_0}}, T_0) = (\underline{\underline{\emptyset}}, (0, y_2^*] \cap \mathbb{X}) \Longrightarrow (S_1, T_1) \Longrightarrow (S_2, T_2)$$
$$\Longrightarrow \cdots$$
$$\Longrightarrow (S_\infty, \underline{\underline{T_\infty}}) = ((0, y_1^*] \cap \mathbb{X}, \underline{\underline{\emptyset}}) \in \widehat{\mathcal{E}},$$

for some $y_1^*, y_2^* \in [0, \infty) \cap \mathbb{X}$.

- **Message:** "*significantly less patient*" $\Longrightarrow$ coercer is coerced!

THANK YOU!!

- *"A Time-Inconsistent Dynkin Game: from Intra-personal to Inter-personal Equilibria"*
  (H. and Z. Zhou), to appear in Finance & Stochastics, available @ arXiv:2101.00343.