Department of Applied Mathematics

Preliminary Examination in Numerical Analysis

August 19, 2019 , 10 am – 1 pm.

Submit solutions to four (and no more) of the following six problems. Show all your work, and justify all your answers. Start each problem on a new page, and write on one side only. No calculators allowed.
Do not write your name on your exam. Instead, write your student number on each page.

Problem 1. Root finding

Consider a random variable X with continuously-differentiable probability density $p(x) > 0$. The cumulative distribution function (cdf) is $F(x) = \int_{-\infty}^{x} p(t)dt.$ Note that $\lim_{x\to\infty} F(x) = 1.$

(a) Show that $F(x)$ is invertible, i.e., $F^{-1}(y)$ exists for $y \in (0,1).$

(b) Let $x = F^{-1}(y).$ Write Newton's method to solve for x given $y \in (0,1)$ using only evaluations of the functions $p(x)$ and $F(x).$

(c) Explain why the method is locally at least quadratically convergent for every $y \in (0,1).$

Problem 2. Quadrature

Consider the integral $\int_0^\infty f(x)dx$ where f is continuous, $f'(0) \neq 0$, and $f(x)$ decays like $x^{-1-\alpha}$ with $\alpha > 0$ in the limit $x \to \infty.$

(a) Suppose you apply the equispaced composite trapezoid rule with n subintervals to approximate $\int_0^L f(x)dx$. What is the asymptotic error formula for the error in the limit $n \to \infty$ with L fixed?

(b) Suppose you consider the quadrature from (a) to be an approximation to the full integral from 0 to ∞. How should L increase with n to optimize the asymptotic rate of total error decay? What is the rate of error decrease with this choice of L?

(c) Make the following change of variable $x = L(1+y)/(1-y),\ \ y = (x-L)/(x+L)$ in the original integral to obtain $\int_{-1}^{1} F_L(y)dy$. Suppose you apply the equispaced composite trapezoid rule; what is the asymptotic error formula for fixed L?

(d) Depending on α, which method - domain truncation or change-of-variable - is preferable?

Problem 3. Linear algebra

(a) Given two self-adjoint (Hermitian) matrices, A and B, where B is a positive (or negative) definite matrix, show that the spectrum of the product of such matrices, AB, is real.

(b) Using 2×2 matrices, construct an example where the product of two real symmetric matrices does not have real eigenvalues.

Problem 4. Interpolation / Approximation

Let function $f \in C^{n+1}[a,b]$, $|f^{(n+1)}(x)| \le M$ and $E_n(f)$ be the error of its best approximation by a polynomial of degree *n*. Show that the accuracy of the best polynomial approximation improves rapidly as the size of the interval $[a,b]$ shrinks, i.e., show that

$$E_n(f) \le \frac{2M}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}.$$

Hint: Use the Chebyshev nodes $x_\ell = \tfrac{1}{2}(b+a)+\tfrac{1}{2}(b-a)\cos\left(\tfrac{\pi}{2}\tfrac{2\ell+1}{n+1}\right)$ to construct a polynomial approximation of *f*.

Problem 5. Numerical ODE

There exists a one parameter family of 2-stage, second order Runge Kutta methods for solving the ODE $y' = f(x, y(x))$. With step size *h* in the *x*-direction, and the parameter α arbitrary, these can be written as

$$\begin{cases} d^{(1)} = h\, f(x_n, y_n) \\ d^{(2)} = h\, f(x_n + \alpha h, y_n + \alpha d^{(1)}) \end{cases}$$

$$y_{n+1} = y_n + \left(1 - \frac{1}{2\alpha}\right) d^{(1)} + \frac{1}{2\alpha} d^{(2)}$$

(a) Verify that these schemes, for all values of α, indeed provide second order accuracy.

Hint: Recall that $y'(x) = f(x, y(x))$, by the chain rule, implies $y''(x) = \frac{\partial f}{\partial x} + f\,\frac{\partial f}{\partial y}$.

(b) Show that these schemes, for all α, have exactly the same stability domain.

(c) Verify that this domain, along the negative real axis, extends exactly over the interval [-2,0].

Problem 6. Numerical PDE

(a) Verify that the PDE $\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3}$ is well posed as an initial value problem.

(b) Consider solving it numerically using the scheme

$$\frac{u(t+k,x) - u(t-k,x)}{2k} = \frac{-\tfrac{1}{2}u(x-2h,t) + u(x-h,t) - u(x+h,t) + \tfrac{1}{2}u(x+2h,t)}{h^3}.$$

Determine this scheme's stability condition.

Hint: The 2π -periodic function $f(\theta) = 2\sin\theta\,(1+\cos\theta)$ oscillates in between $\pm\frac{3\sqrt{3}}{2}$.