DEPARTMENT OF APPLIED MATHEMATICS

PRELIMINARY EXAMINATION IN NUMERICAL ANALYSIS

JANUARY, 2023

Submit solutions to four (and no more) of the following six problems. Justify all your answers.

Root finding

Consider a function $f \in C^{\infty}$ in a neighborhood of its root $f(x^*) = 0$.

1. Show that if x^* is a simple root, i.e. $f'(x^*) \neq 0$, then the Newton's method converges quadratically in some neighborhood of x^*.

2. Determine the rate of convergence of the Newton's method if the root has multiplicity m, where $m \geq 2$.

Numerical Quadrature

Consider a subspace $\mathscr{P}_{N-1}$ of polynomials of degree $N-1$ on the interval $[-1, 1]$ and associated inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx. \tag{0.1}$$

(1) Using the minimal number of nodes, construct a discrete inner product identical to (0.1) on $\mathscr{P}_{N-1}$. Prove that the two inner products coincide on $\mathscr{P}_{N-1}$.

(2) Construct an orthonormal basis in $\mathscr{P}_{N-1}$ such that in this basis the coefficients of expansion of polynomials in $\mathscr{P}_{N-1}$ are given by the scaled values of these polynomials at the nodes. (Hint: use in your construction the Lagrange interpolating polynomials and the discrete inner product).

Interpolation/Approximation

(1) Show that the function

$$\frac{e^{i\theta} - \overline{\alpha}}{1 - \alpha e^{i\theta}}, \ \ \alpha \neq 0,$$

is a unimodular (Blaschke) factor, i.e.

$$\left| \frac{e^{i\theta} - \overline{\alpha}}{1 - \alpha e^{i\theta}} \right| = 1 \text{ for } \ \theta \in [0, 2\pi]. \tag{0.2}$$

(2) Use (0.2) to show that for $|\alpha| < 1$

$$\left\| \left(1 - \alpha e^{i\theta}\right)^{-1} - \sum_{j=0}^{N-1} \alpha^j e^{ij\theta} - \frac{\alpha^N}{1 - \alpha\overline{\alpha}} e^{iN\theta} \right\|_{\infty} = \frac{|\alpha|^{N+1}}{1 - \alpha\overline{\alpha}}$$

Linear Algebra

(1) Consider two $n \times n$ matrices A and B where A is not singular. Show that the eigenvalues of AB and BA are the same.

(2) The trace of a matrix is defined as $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$, where $A = \left\{a_{ij}\right\}_{i,j=1}^{n}$. Show that

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$

and that the trace is invariant under a similarity transform,

$$\mathrm{Tr}\left(X^{-1}AX\right) = \mathrm{Tr}(A).$$

ODEs

Show that the backward differentiation formula,

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}\,h\,f\left(t_{n+2}, y_{n+2}\right),$$

gives rise to a convergent method. State and use appropriate theorems to solve this problem. What is the order of this method?

PDEs

Consider two initial value problems: one for the heat equation,

$$u_t = u_{xx},$$

and, another, for the wave equation written as a first order system,

$$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

both with the periodic boundary conditions. The spatial discretization of these equations leads to a system of ODEs. For the discretization in time, make choice between the following two ODE methods (if you can justify using both, please do so). Your choice is between the implicit midpoint method

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}\left(x_n + \frac{h}{2}, \frac{1}{2}\left(\mathbf{y}_n + \mathbf{y}_{n+1}\right)\right).$$

and the explicit midpoint method (the so-called leap-frog method),

$$\mathbf{y}_{n+2} = \mathbf{y}_n + 2h\mathbf{f}\left(x_{n+1}, \mathbf{y}_{n+1}\right).$$

Provide full justification of the merits of your selection.