Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 2022

Instructions. You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. All problems have equal value.

Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted.
Write your student ID number (not your name!) on your exam.

Problem 1: Root finding

Consider the root finding problem $f(x) = 0$ for $f \in C^2(\mathbb{R})$. For an initial guess x_0 near the single root α, consider the iteration scheme:

$$x_{k+1} = x_k - \gamma_0 f(x_k)$$

Where $\gamma_0 = \frac{1}{f'(x_0)}$. Sometimes this method is called the *chord iteration.*

(a) Rewrite the chord iteration in terms of a fixed point iteration for a function $g(x)$. State necessary conditions (in terms of $f(x)$ and x_0) for the convergence of this scheme to the root $x = \alpha$ of $f(x)$.

(b) Consider the function $f(x) = \sin(3\pi x) + 8x - 4$. One can show it has a unique, simple root at $\alpha \simeq 0.58607509$. Find a set of initial points x_0 for which the chord iteration, if it converges to α, does so *quadratically.*

(c) Recall that the Newton iteration for approximating a root of $f(x)$ is given by

$$x_{k+1} = x_k + p_k$$

where $p_k = -f(x)/f'(x)$.

Consider now an *inexact* Newton iteration where the update to the approximation at step k is q_k where q_k is an approximation of $p_k = -f(x)/f'(x)$. (The chord iteration is an example of an inexact Newton iteration.) This iteration is defined by

$$x_{k+1} = x_k + q_k.$$

Write down an expression for the error of the *inexact Newton* method $e_{k+1} = x_{k+1} - \alpha$ at the $(k+1)^{\text{th}}$ step in terms of the error of the (exact) *Newton* method at the $(k+1)^{\text{th}}$ step.

(d) Assume the difference between the updates in the Newton and inexact Newton iterations at step $k+1$ satisfy

$$|p_k - q_k| \leq \eta_k |f(x_k)|$$

where η_k is small; i.e. your approximate update is close to the Newton update.
With this bound and your solution from part (c), derive an upper bound for the error e_{k+1} of the inexact Newton step. Determine the convergence rate of this method.

Problem 2: Quadrature

For functions defined on a closed interval $[0, 1]$, we want to compute the definite integral

$$I[f] = \int_0^1 f(x) \log(1/x) dx$$

that is, with a logarithmic weight function $\log(1/x) = -\log(x)$. It is possible to find a basis of orthogonal polynomials $P_n(x)$ for the corresponding weighted inner product, and use them to derive n-point Gaussian quadratures with nodes x_k^n and weights ω_k^n. In this problem, you will explore these polynomials and techniques for creating the corresponding Gaussian quadrature.

(a) Let $P_0(x) = 1$. Use Gram-Schmidt or another technique to find $P_1(x)$. Find the corresponding quadrature node x_1^1 and weight ω_1^1 for the 1-point Gaussian quadrature rule.

You may use the formula $\int_0^1 x^m \log(1/x) dx = \frac{1}{(m+1)^2}$.

(b) A general result for orthogonal polynomials is the *interlacing theorem*: considering the partition of $[a, b]$ into $n+1$ subintervals with endpoints $a < x_n^1 < x_n^2 \cdots < x_n^n < b$, there is a unique node for the $n+1$ rule on each of these intervals.

Based on this information, state a quadratically convergent algorithm to find the quadrature nodes for the $(n+1)$ rule, given accurate nodes for the n point rule.

(c) The family P_n satisfies a recursion formula of the form:

$$P_{n+1}(x) = (x - \alpha_n) P_n(x) - \beta_n P_{n-1}(x)$$

Consider the normalized family $Q_n(x) = P_n(x)/\sqrt{h_n}$, with $h_n = \int_0^1 P_n(x)^2 \log(1/x) dx$. Given that $\beta_n = h_n / h_{n-1}$, find a recursive formula for $Q_n(x)$ of the form:

$$\sqrt{\beta_{n+1}} Q_{n+1}(x) = (x - \alpha_n) Q_n(x) - \sqrt{\beta_n} Q_{n-1}(x)$$

(d) Consider the recursive formula above for $n = 0, 1, 2, 3$. Show that $x = \lambda$ is a node of the 4 point Gaussian quadrature if and only if it is an eigenvalue of a symmetric, tridiagonal matrix with diagonal $[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ and super / sub diagonal $[\sqrt{\beta_1}, \sqrt{\beta_2}, \sqrt{\beta_3}]$.

Indicate why deriving this eigenvalue problem using the normalized polynomials might be preferable to solving the eigenvalue problem with the polynomials from the original recursive formula.

Problem 3: Interpolation / Approximation

The Chebyshev polynomials of the third kind, sometimes known as airfoil polynomials, are defined as

$$V_n(x) = \frac{\cos(n + \frac{1}{2})\theta}{\cos\frac{\theta}{2}}$$

$\cos(n - \frac{1}{2} + 1)\theta$

where $x = \cos\theta$, n non-negative integer.

There is a table of trigonometric identities provided at the end of this problem for your convenience.

$x \cos(n - \frac{1}{2}) - \sqrt{1-x^2} \sin$

(a) Show that the Chebyshev polynomials of the third kind satisfy the recursion:

$$V_0(x) = 1$$
$$V_1(x) = 2x - 1$$
$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x)$$

(b) Show that these polynomials are an orthogonal basis for polynomials in $[-1, 1]$ with respect to the inner product:

$$< f, g >= \int_{-1}^{1} f(x)g(x)\sqrt{\frac{1+x}{1-x}}dx$$

That is, with weight function $w(x) = \sqrt{\frac{1+x}{1-x}}$.

Hint: Decide which representation of the polynomials will make the integrals the easiest.

(c) Let $q(x) = \sum_{k=0}^{2} C_k V_k(x)$ be the polynomial that minimizes

$$\min \int_{-1}^{1} (e^x - p(x))^2 \sqrt{\frac{1+x}{1-x}}dx$$

over the space of all polynomials of degree ≤ 2.

Using the results from part (b) derive explicit formulas for the coefficients C_0, C_1, C_2.
You do not have to evaluate the formulas for the coefficients.

Table of trigonometric identities

$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$
$\sin^2(\theta/2) = (1 - \cos\theta)/2$
$\cos^2(\theta/2) = (1 + \cos\theta)/2$
$\tan^2(\theta/2) = (1 - \cos\theta)/(1 + \cos\theta)$
$\cos(\alpha)\cos(\beta) = \frac{1}{2}(\cos(\alpha + \beta) + \cos(\alpha - \beta))$
$\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$

Problem 4: Linear algebra

(a) State and apply the Gerschgorin theorem to the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \epsilon & \epsilon \\ \epsilon & 2 & \epsilon \\ \epsilon & \epsilon & 2 \end{bmatrix}$$

with $\epsilon << 1$.

It is often possible to improve the Gerschgorin bounds on the eigenvalue estimate by first applying a similarity transformation to the matrix $\mathbf{A}$ involving a diagonal matrix $\mathbf{D}_n = \text{diag}(d_1, \ldots, d_n)$.

(b) Prove that a reduction in the error bound of one eigenvalue for the matrix above must occur at the expense of relaxing the bounds for the remaining eigenvalues.

(c) Let $\mathbf{D}_3 = \text{diag}(1, k\epsilon, k\epsilon)$, show that the bounding radius for $\lambda_1 \sim 1$ can be reduced from $\rho_1 = 2\epsilon$ to $2\epsilon^2$.

(d) How would you use the Gerschgorin circle for $\lambda \sim 1$ to compute a numerical approximation for the eigenvalue in that ball?

Problem 5: Numerical ODE

(a) Derive the coefficients c_0, c_1 and c_2 for the backward differentiation formula

$$c_0 u_{n+1} + c_1 u_n + c_2 u_{n-1}$$

which has a truncation error that is second order for approximating $u'(t_{n+1})$.
You must derive the truncation error for this problem.
Be mindful of where you are approximating the derivative.

(b) Show that applying this scheme to approximate the solution of the ODE

$$u' = f(t, u)$$

is convergent.

For the remainder of this problem consider a stiff linear system of ODEs of the form

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0 \tag{1}$$

where $\mathbf{A}$ is a negative definite matrix.

(c) Explain what is meant by a stiff system.

(d) Determine if the scheme derived in part (a) is suitable for solving the stiff system of ODEs (1).

Hint: You don't have to derive the entire region stability domain. It suffices to establish stability for the part of the complex plane that is relevant for the system in (1).

Problem 6: Numerical PDE
Consider the advection equation

$$u_t + u_x = 0 \tag{2}$$

with periodic boundary conditions.

(a) Consider solving (2) by first discretizing u_x in space via the centered difference approximation

$$u_x \approx \frac{u(x+h,t) - u(x-h,t)}{2h}.$$

Using this discretization, write down the matrix $\mathbf{A}$ such that (2) can be written as

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}. \tag{3}$$

(b) Consider solving the ODE system (3) using explicit Euler in time. Using properties of **A**, show that this scheme is unsuitable for solving (2). Explain what the failure mechanism is. (You may state the domain for explicit Euler without proof.)

(c) Consider solving (2) using the discretization

$$\frac{u_m^{n+1} - u_m^{n-1}}{2k} + \frac{u_{m+1}^n - u_{m-1}^n}{2h} = 0$$

where $u_m^n = u(mh, nk)$, k denotes the time step size and h denotes the space step size. Use von Neumann analysis to determine the amplification factor and the region of stability.