Newton's method for 2-point ODE BVP

Consider the IV problem

$$y''(x,s) = f(x, y(x,s), y'(x,s)) \quad ; \quad y(a,s) = \alpha \ , \ y'(a,s) = s$$

(Emphasize in notation the s-dependence)

To apply Newton, we need to calculate $\underbrace{\left.\frac{dy}{ds}\right|_{x=b}}$.

Idea: 1, Introduce $z(x,s) = \frac{dy(x,s)}{ds}$. This $\downarrow$ is then $z(b,s)$.

2, By differentiating the ODE wrt. s, get an ODE for $z(x,s)$. Solve that IVP to obtain $z(b,s)$.

Carry out:

Original ODE: $y''(x,s) = f(x, y(x,s), y'(x,s))$

Its s-derivative: $\frac{d}{ds}[y''(x,s)] = \underbrace{\frac{\partial}{\partial s} f(x, y(x,s), y'(x,s))}_{\frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial y'} \cdot \frac{\partial y'}{\partial s}}$

So ODE for $z(x,s)$: $z''(x,s) = f_y \cdot z + f_{y'} \cdot z'$

Its ICs:

$$\begin{cases} z(a,s) = \frac{\partial}{\partial s} y(x,s)\Big|_{x=a} = 0 \\ z'(a,s) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial s} y(x,s)\right]\Big|_{x=a} = \frac{\partial}{\partial s}\left[\frac{\partial}{\partial x} y(x,s)\right]\Big|_{x=a} = \frac{\partial}{\partial s}[s] = 1. \end{cases}$$

Procedure:

Let s_0 be an initial guess; for ex. $s_0 = \frac{\beta - \alpha}{b-a}$.

Then iterate

$$s_{k+1} = s_k + \frac{\beta - y(b, s_k)}{z(b, s_k)}$$

Note: Solve together for y and z across $[a,b]$ as one coupled system of 4 first order ODEs.