Newton's method for 2-point ODE BVP

Consider the IV problem

\[ y''(x,s) = f(x,y(x,s),y'(x,s)) \quad \text{; } \quad y(a,s) = \alpha \quad \text{; } \quad y'(a,s) = \beta \]

(Emphasize in notation the s-dependence.)

To apply Newton, we need to calculate \( \frac{dy}{ds} \bigr|_{x=a} \).

Idea 1: Introduce \( z(x,s) = \frac{dy(x,s)}{ds} \). This \( \beta \) is then \( z(b,s) \).

2. By differentiating the ODE wrt. \( s \), get an ODE for \( z(x,s) \). Solve that IVP to obtain \( z(b,s) \).

Carryout:

Original ODE:

\[ y''(x,s) = f(x,y(x,s),y'(x,s)) \]

Its s-derivative:

\[ \frac{d}{ds} [y'(x,s)] = \frac{\partial f}{\partial y} \frac{dy}{ds} + \frac{\partial f}{\partial y'} \frac{dy'}{ds} \]

So ODE for \( z(x,s) \):

\[ z''(x,s) = f_y \cdot z + f_{y'} \cdot z' \]

Its ICs:

\[ z(a,s) = \left. \frac{dy}{ds} \right|_{x=a} = 0 \]

\[ z'(a,s) = \left. \frac{d}{dx} \left[ \frac{\partial}{\partial s} y(x,s) \right] \right|_{x=a} = \left. \frac{\partial}{\partial s} \left[ \frac{\partial}{\partial x} y(x,s) \right] \right|_{x=a} = \frac{\partial}{\partial s} \left[ \frac{\partial}{\partial x} y(x,s) \right] \right|_{x=a} = 1 \]

Procedure:

Let \( S_0 \) be an initial guess; for ex. \( S_0 = \frac{\beta - \alpha}{b-a} \).

Then iterate

\[ S_{n+1} = S_n + \frac{\beta - y(b,S_n)}{z(b,S_n)} \]

Note: Solve together for \( y \) and \( z \) across \([a,b]\) as one coupled system of 4 first-order ODEs.