Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
August 2024

Instructions

You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

Problem 1: Root Finding (25 points)

Consider the task of finding the fixed point of the vector function

$$\vec{G}(\vec{x}) = \begin{bmatrix} g_1(\vec{x}) \\ g_2(\vec{x}) \end{bmatrix}$$

where $\vec{x} = (x, y)$ and $g_1(\vec{x})$, $g_2(\vec{x})$ are in $\mathcal{C}^\infty$. Let $\vec{\alpha} = (\alpha_1, \alpha_2)$ denote the fixed point of $\vec{G}(\vec{x})$.

(a) Derive conditions on the function $\vec{G}(\vec{x})$ that gaurantee that the fixed point iteration

$$\vec{x}_{k+1} = \vec{G}(\vec{x}_k)$$

converges to the fixed point $\vec{\alpha}$ for all initial guesses $\vec{x}_0$ in a neighborhood D of the fixed point.

(b) Prove that when the condition you found in part (a) is satisified the fixed point iteration converges linearly.

Problem 2: Interpolation/Approximation (25 points)

We denote the weighted L^2 inner-product of two functions u and v by:

$$(u, v) = \int_a^b w(x)u(x)v(x)dx$$

where w is a positive weight function. We associate to this inner-product the norm

$$\|u\|_{L^2(a,b)} = \left(\int_a^b w(x)(u(x))^2 dx \right)^{1/2}$$

(a) Let $\{\Psi_j\}_{0 \leq j \leq n}$ be a set of nonzero, orthogonal (with respect to $(\cdot, \cdot)$) polynomials of degree less than or equal to n.

The space of polynomials of degree less than or equal to n is denoted by $\mathbb{P}_n$. Prove that the $(\Psi_0, \Psi_1, \ldots, \Psi_n)$ forms a basis for $\mathbb{P}_n$.

(b) Let Φ_j's be a set of polynomials defined by

$$\Phi_0(x) = 1, \quad \Phi_1(x) = x - \frac{(x, 1)}{(1, 1)}$$
$$\Phi_j(x) = (x - a)\Phi_{j-1}(x) - b\Phi_{j-2}(x), \quad j \geq 2$$

with

$$a = \frac{(x\Phi_{j-1}(x), \Phi_{j-1}(x))}{(\Phi_{j-1}(x), \Phi_{j-1}(x))} \quad b = \frac{(x\Phi_{j-1}(x), \Phi_{j-2}(x))}{(\Phi_{j-2}(x), \Phi_{j-2}(x))}$$

Show that the Φ_j's form a set of orthogonal polynomials.

(c) Let $f \in \mathcal{C}(a,b)$. Use orthogonal polynomials to derive a general solution to the following problem

$$\min_{p \in \mathbb{P}_n} \|f - p\|_{L^2(a,b)}$$

(d) Find the line that best approximates $\sqrt{x}$ in the weighted L^2 norm on the interval $(0,1)$. The weight function is chosen to be $w = 1$.

Problem 3: Quadrature (25 points)

Consider the task of numerically approximating

$$I(f) = \int_a^b f(x)dx$$

where $f \in C^\infty[a,b]$.

(a) Derive the trapezoidal rule and corresponding error for approximating $I(f)$.
Useful information: $\int_a^b (x-a)(x-b)dx = -\frac{1}{6}(b-a)^3$

(b) Find the formula for the composite trapezoidal rule using uniform intervals of size $h = \frac{b-a}{n}$ where $n+1$ is the number of quadrature points. i.e. the quadrature points are $x_j = a + j * h$ for $j = 0, \ldots, n$

(c) Derive the error for the composite trapezoidal rule.

Problem 4: Linear Algebra (25 points)

Let $\mathtt{A} \in \mathbb{R}^{n \times n}$ a normal matrix with eigenvalues λ_i and corresponding eigenvectors $\vec{u}_i$ (forming an orthonormal basis of $\mathbb{C}^n$). Further, assume that $|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$. Note that this assumption implies $(\lambda_1, \vec{u}_1)$ are real. We consider the power method to obtain the dominant eigenpair $(\lambda_1, \vec{u}_1)$, given an initial guess for the eigenvector $\vec{z}_0$:

$$\begin{aligned}
\vec{w}_{k+1} &= \mathtt{A}\vec{z}_k \\
\vec{z}_{k+1} &= \frac{\vec{w}_{k+1}}{||\vec{w}_{k+1}||_\infty} \\
\lambda_{k+1} &= \frac{\vec{z}_k^{\,*}\vec{w}_{k+1}}{\vec{z_k}^{\,*}\vec{z_k}}
\end{aligned}$$

(a) Consider the Rayleigh quotient function $R_{\mathtt{A}}(\vec{z}) = \frac{\vec{z}^*\mathtt{A}\vec{z}}{\vec{z}^*\vec{z}}$ for $\vec{z} \in \mathbb{C}^n$. Find a formula for $R_{\mathtt{A}}(\vec{z})$ that does not involve any vectors. Use this formula to prove that the approximation for λ_{k+1} in the power method converges to λ_1 as $k \to \infty$.

(b) Let μ be an estimate for the simple, real eigenvalue λ_q. State a necessary and sufficient condition for μ such that the power method applied to $(\mathtt{A} - \mu\mathtt{I})^{-1}$ converge linearly to $\vec{u}_q$ (up to a scalar factor). What is the linear rate of convergence?

(c) Given $\mu_0 = \mu$ and $\vec{z}_0$, consider the following algorithm:

$$\begin{aligned}
\vec{w}_{k+1} &= (\mathtt{A} - \mu_k \mathtt{I})^{-1} \vec{z}_k \\
\vec{z}_{k+1} &= \frac{\vec{w}_{k+1}}{||\vec{w}_{k+1}||_\infty} \\
\lambda_{k+1} &= \frac{\vec{z}_k^* \mathtt{A} \vec{z}_k}{\vec{z}_k^{\,*} \vec{z}_k} \\
\mu_{k+1} &= \lambda_{k+1}
\end{aligned}$$

Explain in what sense this algorithm is an acceleration of the one proposed in (b) (Hint: does the order or rate improve?).

Problem 5: Numerical ODE (25 points)

We wish to numerically solve the IVP for a system of N first order ODEs $\{\vec{y}'(t) = f(t, \vec{y}),\ \vec{y}(0) = \vec{y}_0\}$. Consider the family of single-step methods

$$y_{n+1} = y_n + \Delta t \left(\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right)$$

for a parameter $\theta \in [0, 1]$. This is sometimes known as the family of θ methods.

(a) Determine for which values of θ these methods are consistent. For the values of θ where the method is consistent, determine the order of the method.

(b) Derive an equation for the region of absolute stability R_θ that lies in the complex plane $\mathbb{C}$. For all values of $\theta \in [0, 1]$, describe geometrically the region of absolute stability. (It looks different for different values of θ. Determine for what values of θ is the method A-stable.

(c) Determine for which values of θ these methods are explicit or implicit. In the case of θ where the method is implicit, explain what method (and what inputs or parameters) you would use to compute the next timestep and why.

Problem 6: Numerical PDE (25 points)

Consider the following IVP for an advection-diffusion PDE with periodic boundary conditions:

$$\begin{aligned}
u_t(x,t) = b u_{xx}(x,t) - a u_x(x,t) + f(x) \quad & (x,t) \in (-\pi, \pi) \times (0, T) \\
u(x, 0) = \phi(x) \quad & x \in (-\pi, \pi) \\
u(-\pi, t) = u(\pi, t) \quad & t \in [0, T]
\end{aligned}$$

with $a \in \mathbb{R}, b \geq 0$ and ϕ, f smooth and 2π periodic.

(a) For a regular grid in x with n points spaced by $\Delta x = \frac{2\pi}{n}$, we can write $U_j(t) \simeq u(x_j, t)$. Using the forward difference for u_x and centered second difference for u_{xx}, write down a system of n first-order ODEs in time for $U_j(t)$.

(b) Note that this system can be written as $\vec{U}(t)' = \mathtt{M}\vec{U}(t) + \vec{F}$. Describe the entries and structure of matrix $\mathtt{M}$.

(c) Use the implicit trapezoidal method to discretize the system above in time, and write down an equation to compute $\vec{U}(t_{k+1})$ in terms of $\vec{U}(t_k)$.

(d) If you are given a formula for the eigenvalues $\{\lambda_j(\Delta x)\}_{j=1}^{n}$ of matrix `M`, explain how you could use them to perform a stability analysis on the finite difference scheme described above.