Department of Applied Mathematics
Preliminary Examination in Numerical Analysis
January 2024

Instructions

You have three hours to complete this exam. Submit solutions to four (and no more) of the following six problems. Please start each problem on a new page. You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. Write your student ID number (not your name!) on your exam.

Problem 1: Rootfinding (25 points)

Consider a rootfinding problem for $f \in C^\infty$ near a simple root $f(\alpha) = 0$. We apply a few iterative methods to solve it; each produces iterates x_n with errors defined by $\epsilon_n = x_n - \alpha$.

1. Assume $x_n \to \alpha$. Write down a definition for *order* of convergence of an iterative method.

2. We wish to consider algorithms with iterates of the form:

$$x_{n+1} = x_n - \gamma_n f(x_n)$$

   where γ_n is some function of $x_n, x_{n-1}, \ldots, x_0$. Explain how the Newton-Raphson and Secant methods are both examples; for each, indicate their order of convergence.

3. Let $M(\alpha) = \frac{f''(\alpha)}{2f'(\alpha)}$, and assume $f''(\alpha) \neq 0$. Using an argument similar to the one used for Newton (involving Taylor expansions of $f(x)$ around α), we can conclude that if the secant method is convergent,

$$\epsilon_{n+1} = M(\alpha)\epsilon_n\epsilon_{n-1}$$

   Based on this formula, give intuition as to how we know whether the order is linear or superlinear (Hint: use an example). Derive an equation for the order of convergence p.

Problem 2: Quadrature (25 points)

In a number of applications in probability and statistics, we need to compute integrals of the form

$$E[h(y)] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} h(y) e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy$$

for $h(y) \in C^\infty(\mathbb{R})$. This yields the expectation of $h(y)$ for a normally distributed random variable with mean $\mu \in \mathbb{R}$ and standard deviation $\sigma > 0$.

1. The Gauss-Hermite quadrature is a gaussian quadrature for integrals

$$I[f] = \int_{-\infty}^{\infty} f(x) e^{-x^2} dx$$

   with weight function $w(x) = e^{-x^2}$. Given nodes and weights $\{x_i, w_i\}_{i=1}^n$ for the n-point quadrature, write down an approximation for $E[h(y)]$ (Hint: use a change of variable).

2. Hermite polynomials are an orthonormal basis for the inner product $< f, g >_w = \int_{-\infty}^{\infty} f(x)g(x)w(x)dx$. Given that $H_0(x) = 1$, derive formulas for $H_1(x), H_2(x)$ (e.g. using Gram-Schmidt). You may use the formula

$$\int_{-\infty}^{\infty} x^n e^{-x^2} dx = \frac{((-1)^n + 1)}{2} \Gamma\left(\frac{n+1}{2}\right)$$

   with $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$, $\Gamma(\frac{5}{2}) = \frac{3\sqrt{\pi}}{4}$ (generally, $\Gamma(x+1) = x\Gamma(x)$).

3. Find the nodes and weights for the 2 point Gauss-Hermite quadrature.

Problem 3: Numerical Linear Algebra (25 points)

Let $A \in \mathbb{R}^{n \times n}$, non-singular (invertible) matrix. Let $u, v \in \mathbb{R}^n$; we define the rank 1 perturbation $\widehat{A} = A + uv^T$. The *Sherman-Morrison* formula gives us a formula for $\widehat{A}^{-1}$, if it exists:

$$\widehat{A}^{-1} = (A + uv^T)^{-1} = A^{-1} - \left(\frac{1}{s_A}\right) A^{-1}uv^T A^{-1} \quad \text{with} \quad s_A = 1 + v^T A^{-1} u$$

1. Let $(A + uv^T)x = b$, and define $y = v^T x$. Based on this, write down a a $n + 1 \times n + 1$ linear system of the form

$$M \begin{bmatrix} x \\ y \end{bmatrix} = c, \quad c \in \mathbb{R}^{n+1}$$

   satisfied by x and y.

2. Use Gaussian elimination on the *last row* of M (i.e. so that $M(n + 1, 1 : n)$ becomes zero). Using the resulting linear system, find a necessary and sufficient condition for $\widehat{A}$ to be invertible. You may assume that A^{-1} is available.

3. Assume that A is such that we can solve a system of the form $Ax = b$ in $O(n)$ floating point operations, or flops (e.g. A is tridiagonal). What is then the computational cost to solve a system $\widehat{A}x = \widehat{b}$ using Sherman-Morrison? Indicate computational cost (flops) on each step.

Problem 4: Interpolation/Approximation (25 points)

Obtain the first degree polynomial achieving the minimax approximation for $f(x) = 1/(1 + x)$ on $[0, 1]$. Formulate the theorem describing properties of the minimax error.

Problem 5: Numerical ODE (25 points)

Consider the two step method (Adams-Bashforth)

$$y_{n+2} = y_{n+1} + h\left[\frac{3}{2}f(t_{n+1}, y_{n+1}) - \frac{1}{2}f(t_n, y_n)\right].$$

- Show that it is convergent and find its order. State the relevant theorems.
- Find the interval on the real axis that is a part of the region of absolute stability.

Problem 6: Numerical PDE (25 points)

Consider the heat equation

$$\frac{\partial \phi}{\partial t} = \partial_x^2 \phi,$$

with initial condition

$$\phi|_{t=0} = \phi_0,$$

and periodic boundary conditions on the interval $[0, 1]$. Fully describe the Crank-Nicolson scheme for this problem. Show that the scheme is unconditionally stable.